VI. *The Theory of the Double Gamma Function.*

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§ 1. INTRODUCTION.

THE present paper continues my researches in the theory of gamma functions. Previously to a certain extent I obtained known results by new methods: none of the succeeding investigations however have, I believe, been undertaken or suggested by other mathematicians.

In the first paper* published in the connection I attempted to give a homogeneous theory of the ordinary gamma function, considered from the point of view of WEIERSTRASS' function theory. I introduced a parameter ω, and showed that the theory was subordinate to that of a function satisfying the difference equation

$$f(z+\omega) - f(z) = z^s,$$

s being any complex quantity.

That theory led naturally to the consideration† of the G function, satisfying the difference equation

$$G(z+1) = \Gamma(z)G(z),$$

and substantially a function all of whose properties could be obtained by differentiating the simple gamma function with respect to the parameter.

I next considered‡ an extended function $G(z/\tau)$ satisfying the two functional relations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right)f(z); \qquad f(z+\tau) = \Gamma(z)\,(2\pi)^{\frac{\tau-1}{2}}\,\tau^{-z+\frac{1}{2}}f(z),$$

and reducible to the G function when $\tau = 1$. Several points in that paper suggested the formation of a symmetrical double gamma function, in which τ should be replaced by the quotient of two parameters ω_1 and ω_2. In the present investigation such a function is defined, and its theory developed in, I hope, complete detail. The function is the natural extension to two parameters of the simple gamma function $\Gamma_1(z\,|\,\omega)$.

It is necessary for a complete exposition of the theory to consider the properties of what I propose to call double Bernoullian numbers and functions: functions which are the natural extension to two parameters of the simple Bernoullian functions, considered in Part II. of the earliest paper of the series. Such a theory is developed in Part I. of the present paper.

In Part II. I consider the elementary theory of the double gamma function. It is shown that certain symmetrical modular constants arise as finite terms of asymptotic expansions in a manner exactly analogous to the origin of EULER's constant γ.

Such considerations lead naturally to Part III., in which are deduced from a contour integral, which is a double generalisation of RIEMANN's ζ function, certain noteworthy asymptotic approximations, of which the most important is an extension

* BARNES, "The Theory of the Gamma Function," 'Messenger of Mathematics,' vol. 29, pp. 64–128.

† BARNES, "The Theory of the G Function," 'Quarterly Journal of Mathematics,' vol. 31, pp. 264–314.

‡ BARNES, "Genesis of the Double Gamma Function," 'Proceedings of the London Mathematical Society,' vol. 31, pp. 358–381.

of STIRLING'S theorem. By the aid of this theory it is possible to express the logarithm of the double gamma function and the double gamma modular constants as contour integrals similar to those given in Part III. of the 'Theory of the Gamma Function.'

In Part IV. I consider the multiplication and transformation theories of double gamma functions as well as certain curious integral formulæ, which correspond to RAABE'S theorem for the simple gamma function, and are elementary cases of a general theorem connecting successive similar transcendents of higher orders.

In Part V. the asymptotic expansion of the double gamma function is obtained, and it is shown that the function cannot arise as the solution of a differential equation whose coefficients are more simple transcendents.

There exist similar functions of any number of parameters, and these transcendents I propose to call multiple gamma functions. I reserve the formal expression of their properties for publication elsewhere. I have worked out the theory for double gamma functions independently inasmuch as, the complex variable being two dimensional, there are many points in which a higher analogy breaks down: and also, since many proofs in the higher theory are, in their simplest form, inductive and, to be rigorous, require a knowledge of the theorem for the two simplest cases. Not only so, but in the case of the double gamma functions it is possible to give easily an algebraical theory (such as that worked out in Part II.), which is more simple than if one derived all the formulæ from the fundamental consideration of certain contour integrals.

I append a statement of the notation adopted in this paper, mentioning the place in the present series of investigations where such notation is used for the first time.

Derivation.	Name.	Symbol.	First occurrence.
Algebraic solution of $f(a+\omega) - f(a) = a^n$	Simple Bernoullian function	$S_n(a \mid \omega)$	"Gamma Function," § 11.
$\dfrac{S'_n(o \mid \omega)}{n}$	Simple Bernoullian number	${}_1B_n(\omega)$	"Gamma Function," § 15.
Algebraic solution of $f(a+\omega_1) - f(a) = S_n(a \mid \omega_2) + \dfrac{S'_{n+1}(0 \mid \omega_2)}{n+1}$	Double Bernoullian function	${}_2S_n(a \mid \omega_1, \omega_2)$	For the case of equal parameters: "G Function," § 15. In general: "Double Gamma Function," § 2.
$\dfrac{{}_2S'_n(o \mid \omega_1, \omega_2)}{n}$	Double Bernoullian number	${}_2B_{n+1}(\omega_1, \omega_2)$	"Double Gamma Function," § 7.
$\omega^{\frac{z}{\omega}-1}\,\Gamma\left(\dfrac{z}{\omega}\right)$	Simple gamma function	$\Gamma_1(z \mid \omega)$	"Gamma Function," § 2.

Derivation.	Name.	Symbol.	First occurrence.
$\frac{d^r}{dz^r}\log\Gamma_1(z\,\vert\,\omega)$	Logarithm derivative of simple gamma function	$\psi_1^{(r)}(z\,\vert\,\omega)$	"Gamma Function," § 2.
Solution of $f(z+1)=\Gamma(z)f(z)$	G function	$G(z)$	"G Function," § 3.
Solution of $f(z+1)=\Gamma\left(\frac{z}{\tau}\right)f(z)$	Unsymmetrical double gamma function	$G(z\,\vert\,\tau)$	"Genesis," &c., § 1.
Vide §§ 18–24	Double gamma function	$\Gamma_2(z\,\vert\,\omega_1,\omega_2)$	"Double Gamma Function," § 19.
$\frac{d^r}{dz^r}\log\Gamma_2(z\,\vert\,\omega_1,\omega_2)$	Logarithm derivatives of double gamma function	$\psi_2^{(r)}, (z\,\vert\,\omega_1,\omega_2)$	"Double Gamma Function," § 19.
$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}(-z)^{s-1}dz$	Simple Riemann ζ (zeta) function	$\zeta(s,a,\omega)$	"Gamma Function," § 23.
$\frac{\iota\Gamma(1-s)}{2\pi}e^{2Ms\pi i}\int_L\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$	Double Riemann ζ function	$\zeta_2(s,a\,\vert\,\omega_1,\omega_2)$	For equal parameters: "G Function," § 23. In general: "Double Gamma Function," § 39.
$-\frac{\gamma}{\omega}+\frac{\log\omega}{\omega}$	Simple gamma modular form	$\gamma_{11}(\omega)$	"Gamma Function," § 2.
Finite terms of certain asymptotic limits	Unsymmetrical double gamma modular forms	$C(\tau)$ $D(\tau)$	"Genesis," &c., §§ 3 and 4.
Do. do.	Symmetrical double gamma modular forms	$\gamma_{21}(\omega_1,\omega_2)$ $\gamma_{22}(\omega_1,\omega_2)$	"Double Gamma Function," §§ 21 and 23.
Do. do.	Glaisher-Kinkelin constant	A	"G Function," § 3.
$\sqrt{\frac{2\pi}{\omega}}$	Simple Stirling modular form	$\rho_1(\omega)$	"Gamma Function," § 31.
Limit of a certain definite integral	Double Stirling modular form	$\rho_2(\omega_1,\omega_2)$	"Double Gamma Function," § 43.
Constants which take the values $0_1 \pm 1$, according to the distribution of ω_1 and ω_2	. . . .	m, m' } M	"Double Gamma Function," § 21. "Double Gamma Function," § 39.

The symbolic notation by which $F_2[f(z+\omega)]$ is written for

$$f(z+\omega_1+\omega_2)-f(z+\omega_1)-f(z+\omega_2)$$

is introduced in § 49.

PART I.

The Theory of Double Bernoullian Functions and Numbers.

§ 2. In the "Theory of the Gamma Function," Part II., we have defined the simple Bernoullian function $S_n(a\,|\,\omega)$ as that solution of the difference equation

$$f(a+\omega) - f(a) = a^n,$$

where n is a positive integer, which is such that it is an algebraical polynomial and $S_n(o\,|\,\omega) = 0$. And it was proved that such a solution does exist.

In exactly the same manner it may be proved that the difference equation

$$f(a+\omega_1) - f(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1}$$

has an algebraic solution, which is a rational integral polynomial of degree $n+2$.

The difference between any two solutions will be a simply periodic function of period ω, and will therefore be a constant if the solutions are both algebraic polynomials.

There thus exists a unique algebraical polynomial of degree $n+2$, which is a solution of the difference equation.

$$f(a+\omega_1) - f(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1},$$

with the condition $f(o) = 0$.

This solution we call the double Bernoullian function of a with parameters ω_1 and ω_2, and we denote it by ${}_2S_n(a\,|\,\omega_1,\omega_2)$. By symmetry with this notation the simple Bernoullian function would be denoted by ${}_1S_n(a\,|\,\omega)$.

We shall often omit the parameters ω_1 and ω_2, when there is no doubt as to their existence, and write the function simply ${}_2S_n(a)$.

§ 3. We now proceed to show that the double Bernoullian function of a of order n is also the unique algebraical polynomial which is the solution of the difference equation

$$f(a+\omega_2) - f(a) = S_n(a\,|\,\omega_1) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1},$$

with the condition $f(o) = 0$.

For since

$$ {}_2S_n(a+\omega_1) - {}_2S_n(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1} \quad \ldots \ldots \quad (1.)$$

we have at once

$$ {}_2S_n(a+\omega_1+\omega_2) - {}_2S_n(a+\omega_2) - {}_2S_n(r+\omega_1) + {}_2S_n(a)$$
$$= S_n(a+\omega_2\,|\,\omega_2) - S_n(a\,|\,\omega_2) = a^n = S_n(a+\omega_1\,|\,\omega_1) - S_n(a\,|\,\omega_1).$$

And therefore if we put

$$f_n(a) = {}_2S_n(a + \omega_2) - {}_2S_n(a) - S_n(a \mid \omega_1)$$

we shall have

$$f_n(a + \omega_1) - f_n(a) = 0.$$

Now ${}_2S_n(a)$ and $S_n(a/\omega_1)$ are algebraical polynomials of a, and therefore $f_n(a)$ is also such a function. And, therefore, since it is simply periodic of period ω_1, it must be a constant. Thus we have

$$ {}_2S_n(a + \omega_2) - {}_2S_n(a) = S_n(a \mid \omega_1) + \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Again, integrating the relation (1) with respect to a between o and ω_2, we have

$$\int_0^{\omega_1+\omega_2} {}_2S_n(a)\,da - \int_0^{\omega_1} {}_2S_n(a)da - \int_0^{\omega_2} {}_2S_n(a)da = 0,$$

since

$$\int_0^{\omega_2} S_n(a \mid \omega_2)\,da = -\omega_2 \frac{S'_{n+1}(o \mid \omega_2)}{n+1}.$$

Integrating the relation (2) in the same manner between o and ω_1, we obtain for the value of the constant

$$-\frac{1}{\omega_1}\int_0^{\omega_1} S_n(a \mid \omega_1)\,da = \frac{S'_{n+1}(o \mid \omega_1)}{n+1}.$$

And thus ${}_2S_n(a)$ is the unique algebraic solution of the equation

$$f(a + \omega_2) - f(a) = S_n(a \mid \omega_1) + \frac{S'_{n+1}(o \mid \omega_1)}{n+1},$$

with the condition $f(o) = 0$.

From the symmetrical nature of the equations which give ${}_2S_n(a \mid \omega_1, \omega_2)$, we see that this function itself must be symmetrical in ω_1 and ω_2.

§ 4. If now we assume

$$ {}_2S_n(a \mid \omega_1, \omega_2) = \alpha_{n+2}\, a^{n+2} + \alpha_{n+1}\, a^{n+1} + \ldots + \alpha_1 a,$$

the calculation of the highest coefficients may be readily effected.

For we have

$$ {}_2S_n(a + \omega_1) - {}_2S_n(a) = S_n(a \mid \omega_2) + \frac{S'_{n+1}(o \mid \omega_2)}{n+1}$$

$$= \frac{a^{n+1}}{(n+1)\,\omega_2} - \frac{a^n}{2} + \binom{n}{1}\frac{B_1}{2}\, a^{n-1}\omega_2 - \binom{n}{3}\frac{B_2}{3}\, a^{n-3}\omega_2^{\,3} + \ldots$$

Hence if we substitute the assumed expansion for ${}_2S_n(a)$ and equate corresponding powers of a, we find

$$(n+2)\,\omega_1\,\alpha_{n+2} = \frac{1}{(n+1)\,\omega_2},$$

$$\frac{(n+2)(n+1)}{1\,.\,2}\,\omega_1^{\,2}\,\alpha_{n+2} + (n+1)\,\omega_1\,\alpha_{n+1} = -\tfrac{1}{2},$$

$$\frac{(n+2)(n+1)n}{3!}\omega_1^3\,\alpha_{n+2} + \frac{(n+1)n}{2!}\omega_1^2\,\alpha_{n+1} + n\omega_1\,\alpha_n = \binom{n}{1}\frac{B_1}{2}\omega_2,$$

and so on.

On solving these equations successively we readily obtain

$$\alpha_{n+2} = \frac{1}{(n+1)(n+2)\omega_1\omega_2},$$

$$\alpha_{n+1} = -\frac{\omega_1+\omega_2}{2(n+1)\omega_1\omega_2},$$

and, since $B_1 = \frac{1}{6}$,

$$\alpha_n = \frac{1}{12}\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{\omega_1\omega_2}.$$

Thus

$${}_2S_n(a\,|\,\omega_1, \omega_2) = \frac{a^{n+2}}{(n+1)(n+2)\omega_1\omega_2} - \frac{a^{n+1}(\omega_1+\omega_2)}{2(n+1)\omega_1\omega_2} + a^n\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2} + \ldots$$

Further terms can be calculated if necessary. It will be seen, however, that they form what we propose to call double Bernoullian numbers, whose properties may be investigated without the necessity of their formal evaluation.

Corollary. We note that

$${}_2S_1(a\,|\,\omega_1, \omega_2) = \frac{a^3}{6\omega_1\omega_2} - \frac{a^2(\omega_1+\omega_2)}{4\omega_1\omega_2} + a\,\frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2}.$$

And hence

$${}_2S'_1(a\,|\,\omega_1, \omega_2) = \frac{a^2}{2\omega_1\omega_2} - \frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2} + \frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{12\omega_1\omega_2},$$

$${}_2S_1^{(2)}(a\,|\,\omega_1, \omega_2) = \frac{a}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2},$$

$${}_2S_1^{(3)}(a\,|\,\omega_1, \omega_2) = \frac{1}{\omega_1\omega_2}.$$

It will be found that these expressions are of constant occurrence in the course of the present investigation.

Note also that

$${}_2S_0(a\,|\,\omega_1, \omega_2) = \frac{a^2}{2\omega_1\omega_2} - \frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2}.$$

§ 5. We will now prove that, if $n-k \gtrless 0$, $k > 0$,

$${}_2S_n^{(k)}(a\,|\,\omega_1, \omega_2) = \frac{n!}{(n-k)!}\,{}_2S_{n-k}(a\,|\,\omega_1, \omega_2) + {}_2S_n^{(k)}(o\,|\,\omega_1, \omega_2).$$

We have, when $n-k \gtrless 0$ and $k < 0$,

$$\begin{aligned}{}_2S_n^{(k)}(a+\omega_1) - {}_2S_n^{(k)}(a) &= S_n^{(k)}(a\,|\,\omega_2)\\ &= S_n^{(k)}(o\,|\,\omega_2) + \frac{n!}{(n-k)!}S_{n-k}(a\,|\,\omega_2)\end{aligned}$$

("Theory of the Gamma Function," § 14).

and therefore

$$
\begin{aligned}
&{}_2S_n^{(k)}(a+\omega_1) - {}_2S_n^{(k)}(a) \\
&= S_n^{(k)}(o \mid \omega_2) + \frac{n!}{(n-k)!}\left[{}_2S_{n-k}(a+\omega_1) - {}_2S_{n-k}(a) - \frac{S_{n-k+1}(o \mid \omega_2)}{n-k+1}\right].
\end{aligned}
$$

Thus if we write

$$
f(a) = {}_2S_n^{(k)}(a) - \frac{n!}{n-k!}\, {}_2S_{n-k}(a)
$$

we shall have

$$
\begin{aligned}
f(a+\omega_1) - f(a) &= S_n^{(k)}(o \mid \omega_2) - \frac{n!}{(n-k+1)!}\, S'_{n-k+1}(o \mid \omega_2) \\
&= 0 \text{ (“Gamma Function,” § 15).}
\end{aligned}
$$

Similarly

$$
f(a+\omega_2) - f(a) = 0.
$$

and therefore since $f(a)$ is an algebraical polynomial in a, it is a constant.

On making $a = 0$ and remembering that ${}_2S_n(o) = 0$, we obtain

$$
{}_2S_n^{(k)}(a) - \frac{n!}{(n-k)!}\, {}_2S_{n-k}(a) = {}_2S_n^{(k)}(o),
$$

which is the required result.

§ 6. We are now in a position to prove that

$$
\begin{aligned}
\int_0^{\omega_1} {}_2S_n(a)\, da &= -\frac{\omega_1}{n+1}\, {}_2S'_{n+1}(o \mid \omega_1, \omega_2) + \frac{S'_{n+2}(o \mid \omega_2)}{(n+1)\,.\,(n+2)} \\
\int_0^{\omega_2} {}_2S_n(a)\, da &= -\frac{\omega_2}{n+1}\, {}_2S'_{n+1}(o \mid \omega_1, \omega_2) + \frac{S'_{n+2}(o \mid \omega_1)}{(n+1)\,.\,(n+2)}
\end{aligned}
$$

and at the same time the important relation

$$
{}_2S_{n+k}^{(k)}(o \mid \omega_1, \omega_2) = \frac{(n+k)!}{(n+1)}\, {}_2S'_{n+1}(o \mid \omega_1, \omega_2).
$$

Since ${}_2S_n(o) = 0$ we see from the fundamental difference equation that

$$
{}_2S_n(\omega_1) = \frac{S'_{n+1}(o \mid \omega_2)}{n+1}.
$$

Hence, if we put $a = \omega$, in § 5, we see that when $n-k \gtreqless 0$, and $k > 0$,

$$
\begin{aligned}
{}_2S_n^{(k)}(\omega_1) - {}_2S_n^{(k)}(o) &= \frac{n!}{(n-k)!}\, {}_2S_{n-k}(\omega_1) \\
&= \frac{n!}{(n+1-k)!}\, S'_{n-k+1}(o \mid \omega_2).
\end{aligned}
$$

Take now the relation

$$
{}_2S_n^{(k)}(a) = \frac{n!}{(n-k)!}\, {}_2S_{n-k}(a) + {}_2S_n^{(k)}(o),
$$

and integrate with respect to a between 0 and ω_1, we obtain

$$ {}_2S_n^{(k-1)}(\omega_1) - {}_2S_n^{(k-1)}(o) = \frac{n\,!}{(n-k)\,!}\int_0^{\omega_1} {}_2S_{n-k}(a)\,da + \omega_1\,{}_2S_n^{(k)}(o), $$

so that

$$ \frac{n\,!}{(n-k)\,!}\int_0^{\omega_1} {}_2S_{n-k}(a)\,da = -\,\omega_1\,{}_2S_n^{(k)}(o) + \frac{n\,!}{(n+2-k)\,!}\,S'_{n+2-k}(o\,|\,\omega_2). $$

Write now n for $(n-k)$, as is evidently allowable, since both n and k are positive integers, and we have

$$ \int_0^{\omega_1} {}_2S_n(a)\,da = -\,\omega_1 \frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o) + \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1).(n+2)}. $$

We thus see that

$$ \frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o\,|\,\omega_1,\,\omega_2) $$

is independent of k, since this is the only time in the relation just obtained which depends on k.

Putting then $k = 1$, we have when $k > 0$ and $n \gtrless k$.

$$ {}_2S^{(k)}_{n+k}(o\,|\,\omega_1,\,\omega_2) = \frac{(n+k)\,!}{(n+1)\,!}\,{}_2S'_{n+1}(o\,|\,\omega_1,\,\omega_2), $$

which is one of the relations required.

And also

$$ \int_0^{\omega_1} {}_2S_n(a)\,da = -\,\frac{\omega}{n+1}\,{}_2S'_{n+1}(o) + \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1).(n+2)}, $$

another of the given relations. The second integral formula of course may be written down by symmetry.

We notice that in the notation formerly introduced ("Gamma Function," § 15) we have

$$ \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1).(n+2)} = \frac{{}_1B_{n+2}(\omega_2)}{n+2}, $$

and therefore that each of these expressions

$$ = 0 \qquad \text{when } n \text{ is odd,} $$

$$ = \frac{(-)^{\frac{n}{2}} B_{\frac{n}{2}+1}\,\omega_2^{\,n+1}}{(n+1).(n+2)} \qquad \text{when } n \text{ is even.} $$

Thus we see that when n is odd

$$ \int_0^{\omega_1} {}_2S_n(a)\,da = -\,\frac{\omega_1}{n+1}\,{}_2S'_{n+1}(o). $$

§ 7. We now introduce double Bernoullian numbers analogous to the simple Bernoullian numbers introduced in the theory of the gamma function.

In that theory the simple nth Bernoullian number was defined by the relation

$$ {}_1B_n(\omega) = \frac{S'_n(o \mid \omega)}{n}, $$

and now the nth double Bernoullian number is given by

$$ {}_2B_n(\omega_1, \omega_2) = \frac{{}_2S'_n(o \mid \omega_1, \omega_2)}{n}. $$

We note that by the theorem of § 6 we may put

$$ {}_2B_{n+1}(\omega_1, \omega_2) = \frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o \mid \omega_1, \omega_2), $$

and therefore

$$ {}_2S_n^{(k)}(o) = \frac{n\,!}{(n-k)}\,{}_2B_{n-k+1}(\omega_1, \omega_2). $$

§ 8. At this point we may conveniently note the reduction which takes place in the double Bernoullian functions and numbers when the parameters are equal to one another.

If we put $\omega_1 = \omega_2 = \omega$ we have as the single difference equation of the nth double Bernoullian function the relation

$$ f(a+\omega) - f(a) = S_n(a \mid \omega) + {}_1B_{n+1}(\omega), $$

and the function is now defined as the algebraical solution of this equation with the condition ${}_2S_n(o \mid \omega, \omega) = 0$.

Put now

$$ f(a) = -S_{n+1}(a \mid \omega) + (a-\omega)\,S_n(a \mid \omega) + a\,\frac{S'_{n+1}(o \mid \omega)}{n+1}, $$

and we have

$$ f(a+\omega) - f(a) = \omega S_n(a \mid \omega) + \frac{\omega S'_{n+1}(o \mid \omega)}{n+1}. $$

Hence, the other definition conditions being satisfied, we see that

$$ {}_2S_n(a \mid \omega, \omega) = \frac{a-\omega}{\omega}\,S_n(a \mid \omega) - \frac{1}{\omega}\,S_{n+1}(a \mid \omega) + \frac{a}{\omega}\,\frac{S'_{n+1}(o \mid \omega)}{n+1}, $$

that is, the double Bernoullian function when the parameters are equal reduces to simple Bernoullian functions and numbers.

It will be seen later that it is for this reason that it was possible to obtain all the expansions in the theory of the G function in terms of simple Bernoullian functions.

Note that the above relation may be written

$$ {}_2S_n(a \mid \omega, \omega) = \frac{a-\omega}{\omega}\,S_n(a \mid \omega) - \frac{1}{\omega}\,S_{n+1}(a \mid \omega) + \frac{a}{\omega}\,{}_1B_{n+1}(\omega). $$

On differentiation we have

$$ {}_2S'_n(a\,|\,\omega, \omega) = S_n(a\,|\,\omega) + \frac{a-\omega}{\omega} S'_n(a\,|\,\omega) - \frac{1}{\omega} S'_{n+1}(a\,|\,\omega) + \frac{1}{\omega}\,{}_1B_{n+1}(\omega), $$

so that
$$ {}_2S'_n(o\,|\,\omega, \omega) = - S'_n(o\,|\,\omega) - \frac{n}{\omega}\,{}_1B_{n+1}(\omega), $$

and therefore
$$ {}_2B_n(\omega, \omega) = - {}_1B_n(\omega) - \frac{1}{\omega}\,{}_1B_{n+1}(\omega). $$

§ 9. We now see at once that

$$ {}_2S_n(a\,|\,\omega_1, \omega_2) = \frac{a^{n+2}}{(n+1)(n+2)\,\omega_1\,\omega_2} - \frac{\omega_1+\omega_2}{2\,(n+1)\,\omega_1\,\omega_2} a^{n+1} + {}_2B_1(\omega_1, \omega_2)\, a^n $$
$$ + \binom{n}{1}\,{}_2B_2(\omega_1, \omega_2)\, a^{n-1} + \binom{n}{2}\,{}_2B_3(\omega_1, \omega_2)\, a^{n-2} + \ldots $$

and so complete the expansion of § 4.

For by MACLAURIN'S theorem we have

$$ {}_2S_n(a\,|\,\omega_1, \omega_2) = a\,{}_2S'_n(o) + \frac{{}_2S_n^{(2)}(o)}{2\,!} a^2 + \ldots \qquad + \frac{{}_2S_n^{(n+2)}(o)}{(n+2)\,!} a^{n+2}, $$

since the higher differentials vanish.

From the few terms found in § 4 we see that

$$ {}_2S_n^{(n+2)}(o) = \frac{n\,!}{\omega_1\omega_2}, $$
$$ {}_2S_n^{(n+1)}(o) = - \frac{n\,!\,(\omega_1+\omega_2)}{2\omega_1\omega_2}. $$

Now when $n \geqq k$ and $k > 0$, we have

$$ {}_2S_n^{(k)}(o) = \frac{n\,!}{(n-k)\,!}\;{}_2B_{n-k+1}(\omega_1, \omega_2), $$

and thus we have the expansion in question.

§ 10. It may now be shown that

$$ {}_2S_n(a) = (-)^n\,{}_2S_n(\omega_1+\omega_2-a) + \frac{(-)^{n-1}}{n+1}\left[S'_{n+1}(o\,|\,\omega_1) + S'_{n+1}(o\,|\,\omega_2)\right], $$

or, as we may write it,

$$ {}_2S_n(a) = (-)^n\,{}_2S_n(\omega_1+\omega_2-a) + (-)^{n-1}\left[{}_1B_{n+1}(\omega_1) + {}_1B_{n+1}(\omega_2)\right]. $$

Remembering the value of, ${}_1B_{n+1}(\omega)$ we thus prove that

${}_2S_n(a) = {}_2S_n(\omega_1+\omega_2-a)$, when n is even,

${}_2S_n(a) = -\,{}_2S_n(\omega_1+\omega_2-a) + \dfrac{(-)^{\frac{n-1}{2}}}{n+1}\; B_{\frac{n+1}{2}}(\omega_1{}^n + \omega_2{}^n)$, when n is odd.

Substitute $\omega_2 - a$ for a in the fundamental difference equation and we find

$$ {}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_2 - a) = S_n(\omega_2 - a \mid \omega_2) + {}_1B_{n+1}(\omega_2), $$

and therefore, since $S_n(\omega_2 - a \mid \omega_2) = (-)^{n+1} S_n(a \mid \omega_2)$, we have

$$ {}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_2 - a) = (-)^{n-1} S_n(a \mid \omega_2) + {}_1B_{n+1}(\omega_1). $$

If therefore we put

$$ f(a) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2) \right] + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2), $$

we see that $f(a)$ is an algebraic solution of the difference equation

$$ f(a + \omega_1) - f(a) = S_n(a \mid \omega_2) + {}_1B_{n+1}(\omega_2), $$

and therefore can only differ by a constant from ${}_2S_n(a)$.

Determine this constant by making $a = 0$ and we have the relation

$$ {}_2S_n(a) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_1 + \omega_2) \right] + \frac{(-)^n a}{\omega_1}\, {}_1B_{n+1}(\omega_2) $$
$$ + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2). $$

When n is odd the last two terms cancel each other, and when n is even ${}_1B_{n+1}(\omega_2)$ vanishes.

Hence $$ {}_2S_n(a \mid \omega_1, \omega_2) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_1 + \omega_2) \right]. $$

From the fundamental difference equations we see at once that

$$ {}_2S_n(\omega_1 + \omega_2) = \frac{S_{n+1}(o \mid \omega_1)}{n+1} + \frac{S_{n+1}(o \mid \omega_2)}{n+1} $$
$$ = {}_1B_{n+1}(\omega_1) + {}_1B_{n+1}(\omega_2), $$

and therefore we have the relation stated.

§ 11. We may now show that, when n is even,

$$ {}_2B_n(\omega_1, \omega_2) = (-)^{\frac{n}{2}} \frac{B^{n/2}}{2n} (\omega_1^{\,n-1} + \omega_2^{\,n-1}), $$

a simple expression for the even double Bernoullian numbers which corresponds in some degree to the fact that the even simple Bernoullian numbers vanish.

On differentiating with regard to a the result of the previous paragraph we find

$$ {}_2S'_n(a \mid \omega_1, \omega_2) + (-)^n\, {}_2S'_n(\omega_1 + \omega_2 - a) = 0. $$

From the fundamental difference equations we have

$$ {}_2S_n(a + \omega_1 + \omega_2) - {}_2S_n(a + \omega_1) - {}_2S_n(a + \omega_2) + {}_2S_n(a) = a^n, $$

and hence $${}_2S'_n(\omega_1+\omega_2) = {}_2S'_n(\omega_1) + {}_2S'_n(\omega_2) - {}_2S'_n(o)$$
$$= {}_2S'_n(o) + S'_n(o\,|\,\omega_2) + S'_n(o\,|\,\omega_1).$$

Thus, since ("Gamma Function," § 15)

$$S'_n(o\,|\,\omega) = 0 \quad \text{when } n \text{ is odd}$$
$$= (-)^{\frac{n}{2}-1} B_{\frac{n}{2}}\, \omega_1^{n-1} \quad \text{when } n \text{ is even,}$$

we see that $${}_2S'_n(\omega_1+\omega_2) = {}_2S'_n(o), \text{ when } n \text{ is odd;}$$

and $${}_2S'_n(\omega_1+\omega_2) = {}_2S'_n(o) + (-)^{\frac{n}{2}-1} B_{\frac{n}{2}}(\omega_1^{n-1}+\omega_2^{n-1}), \text{ when } n \text{ is even.}$$

But our former relation gives us, when n is even,

$${}_2S'_n(\omega_1+\omega_2) = -\,{}_2S'_n(o).$$

Hence, when n is even,

$${}_2S'_n(o) = (-)^{\frac{n}{2}} \cdot \frac{B_{\frac{n}{2}}}{2} \cdot (\omega_1^{n-1}+\omega_2^{n-1}),$$

which is equivalent to the relation required.

§ 12. We now proceed to show that

$$\int_0^a {}_2S_n(a)\,da = \frac{{}_2S_{n+1}(a)}{n+1} - a\,{}_2B_{n+1}(\omega_1,\omega_2).$$

We have

$${}_2S_n(a+\omega_1) - {}_2S_n(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1}.$$

Hence, integrating with respect to a

$$\int_0^{a+\omega_1} {}_2S_n(a)\,da - \int_0^a {}_2S_n(a)\,da = \int_0^{\omega_1} {}_2S_n(a)\,da + \int_0^a S_n(a\,|\,\omega_2)\,da + a\frac{S'_{n+1}(o\,|\,\omega_2)}{n+1}$$

But ("Gamma Function," § 19)

$$\int_0^a S_n(a\,|\,\omega_2)\,da + a\,\frac{S'_{n+1}(o\,|\,\omega_2)}{n+1} = \frac{S_{n+1}(a\,|\,\omega_2)}{n+1}.$$

so that, if $f(a) = \int_0^a {}_2S_n(a)\,da$, this function is an algebraic solution of the difference equation

$$f(a+\omega_1) - f(a) = \int_0^{\omega_1} {}_2S_n(a)\,da + \frac{S_{n+1}(a\,|\,\omega_2)}{n+1}.$$

But this difference equation is evidently satisfied by

$$\frac{a}{\omega_1}\left[\int_0^{\omega_1} {}_2S_n(a)\,da - \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1)(n+2)}\right] + \frac{1}{n+1}\,{}_2S_{n+1}(a)$$
$$= \frac{a}{\omega_1}\left[-\frac{\omega_1}{n+1}\,{}_2S'_{n+1}(o)\right] + \frac{{}_2S_{n+1}(a)}{n+1} \qquad \text{by § 6,}$$
$$= \frac{{}_2S_{n+1}(a)}{n+1} - a\,{}_2B_{n+1}(\omega_1,\omega_2).$$

Hence as these two solutions both vanish when $a = 0$, we have

$$\int_0^a {}_2S_n(a)\,da = \frac{{}_2S_{n+1}(a)}{n+1} - a\,{}_2B_{n+1}(\omega_1, \omega_2).$$

As a *corollary* we have on differentiation

$${}_2S'_{n+1}(a) = (n+1)\,{}_2S_n(a) + {}_2S'_{n+1}(o).$$

§ 13. The multiplication theory of double Bernoullian functions may be conveniently expressed by the formula

$${}_2S_n(ma \mid \omega_1, \omega_2) = m^n\,{}_2S_n\left(a \mid \frac{\omega_1}{m}, \frac{\omega_2}{m}\right).$$

From the fundamental difference relation we have

$${}_2S_n\left\{m\left(a + \frac{\omega_1}{m}\right) \middle| \omega_1, \omega_2\right\} = {}_2S_n\{ma \mid \omega_1, \omega_2\} + m^n\left[S_n\left(a \middle| \frac{\omega_2}{m}\right) + \frac{S'_{n+1}\left(\frac{\omega_2}{m}\right)}{n+1}\right].$$

Hence $\frac{1}{m^n}\,{}_2S_n(ma \mid \omega_1, \omega_2)$ satisfies the difference equation

$$f\left(a + \frac{\omega_1}{m}\right) - f(a) = S_n\left(a \middle| \frac{\omega_2}{m}\right) + \frac{S'_{n+1}\left(\frac{\omega_2}{m}\right)}{n+1},$$

and is the only algebraical solution such that $f(o) = 0$.

Hence $$\frac{1}{m^n}\,{}_2S_n(ma \mid \omega_1, \omega_2) = {}_2S_n\left(a \middle| \frac{\omega_1}{m}, \frac{\omega_2}{m}\right),$$

which is the relation required.

As a *corollary* we see that the nth double Bernoullian number is homogeneous and of degree $(n-1)$ in the ω's.

For in § 9 we have seen that in the expansion of

$${}_2S_n(a \mid \omega_1, \omega_2),$$

the part of the coefficient of a^{n-s+1} which involves the ω's is ${}_2B_s(\omega_1, \omega_2)$.

§ 14. The transformation of the parameters of the double Bernoullian function is given by the relation

$${}_2S_n\left(a \middle| \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} {}_2S_n\left(a + \frac{k\omega_1}{p} + \frac{l\omega_2}{q} \middle| \omega_1, \omega_2\right) + pq\,{}_2B_{n+1}(\omega_1, \omega_2) - {}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right),$$

as we proceed to prove.

Let $$f(a) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} {}_2S_n\left(a + \frac{k\omega_1}{p} + \frac{l\omega_2}{q} \middle| \omega_1, \omega_2\right),$$

then we obtain at once

$$f\left(a + \frac{\omega_1}{p}\right) - f(a) = \sum_{l=0}^{q-1}\left[{}_2S_n\left(a + \frac{l\omega_2}{q} + \omega_1\right) - {}_2S_n\left(a + \frac{l\omega_2}{q}\right)\right]$$

$$= \sum_{l=0}^{q-1}\left[S\left(a + \frac{l\omega_2}{q}\middle|\omega_2\right) + {}_1B_{n+1}(\omega_2)\right]$$

$$= S_n\left(a\middle|\frac{\omega_2}{q}\right) + {}_1B_{n+1}\left(\frac{\omega_2}{q}\right) \text{ ("Gamma Function," § 18).}$$

Thus $f(a)$ is an algebraic solution of the difference equation satisfied by ${}_2S_n\left(a\middle|\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$. The two solutions can then only differ by a constant, and thus

$${}_2S_n\left(a\middle|\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1} {}_2S_n\left(a + \frac{k\omega_1}{p} + \frac{l\omega_2}{q}\middle|\omega_1, \omega_2\right) + R_n,$$

where R_n is independent of a.

To determine this constant, let us integrate between 0 and $\frac{\omega_1}{p}$. We find

$$\int_0^{\frac{\omega_1}{p}} {}_2S_n\left(a\middle|\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) da = \sum_{l=0}^{q-1}\int_0^{\omega_1} {}_2S_n\left(\zeta + \frac{l\omega_2}{q}\middle|\omega_1, \omega_2\right) d\zeta + \frac{\omega_1}{p} R_n,$$

and therefore by § 6

$$-\frac{\omega_1}{p}\,{}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + \frac{{}_1B_{n+2}\left(\frac{\omega_2}{q}\right)}{n+1},$$

$$= -q\omega_1\,{}_2B_{n+1}(\omega_1, \omega_2) + \sum_{l=0}^{q-1}\frac{{}_2S_{n+1}\left(\omega_1 + \frac{\omega}{p}\right) - {}_2S_{n+1}\left(\frac{l\omega_2}{q}\right)}{n+1} + \frac{\omega_1}{p}R_n, \text{ by § 12.}$$

$$= -q\omega_1\,{}_2B_{n+1}(\omega_1, \omega_2) + \left(\frac{1}{q^{n+1}} - q\right)\frac{{}_1B_{n+2}(\omega_2)}{n+1} + q\,\frac{{}_1B_{n+2}(\omega_2)}{n+1} + \frac{\omega_1}{p}R_n$$

("Gamma Function," § 18).

And therefore

$$-\frac{\omega_1}{p}\,{}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = -q\omega_1\,{}_2B_{n+1}(\omega_1, \omega_2) + \frac{\omega_1}{p}R_n,$$

so that

$$R_n = pq\,{}_2B_{n+1}(\omega_1, \omega_2) - {}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right).$$

On substituting this value we have the theorem enunciated.

As a *corollary* we have on making $a = 0$.

$$\sum_{k=0}^{p-1}\sum_{l=0}^{q-1} {}_2S_n\left(\frac{k\omega_1}{p} + \frac{l\omega_2}{q}\middle|\omega_1, \omega_2\right) = {}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) pq\,{}_2B_{n+1}(\omega_1, \omega_2),$$

§ 15. We now proceed to prove the expansion of fundamental importance in the theory of double Bernoullian functions :—

$$\frac{ze^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} = \frac{{}_2S_1^{(3)}(a)}{z} - {}_2S_1^{(2)}(a) + \frac{{}_2S_1'(a)}{1!}z + \ldots + \frac{(-)^{n-1}{}_2S_n'(a)}{n!}z^n + \ldots$$

In this expansion the double Bernoullian functions have ω_1 and ω_2 for parameters, and the expansion is valid provided $|z|$ is less than the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|, \left|\frac{2\pi\iota}{\omega_2}\right|$.

For within a circle whose radius is less than this quantity, the function

$$\frac{z^2 e^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}$$

has no poles, and hence it is expansible in a Taylor's series of powers of z.

Thus we may assume, for all finite values of a,

$$\frac{ze^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} = \frac{A_1(a)}{z} - A_0(a) + \frac{f_1(a)}{1!} z + \ldots + (-)^{n-1} \frac{f_n(a)}{n!} z^n + \ldots$$

where it is obvious that the functions of a which enter as coefficients are all algebraical polynomials.

Change now a into $a + \omega_1$, and subtract the expansion so obtained from the one just written.

We find

$$\frac{ze^{-az}}{1 - e^{-\omega z}} = \frac{A_1(a) - A_1(a + \omega_1)}{z} - A_0(a) + A_0(a + \omega_1) + \frac{f_1(a) - f_1(a + \omega_1)}{1!} z + \ldots$$

$$\ldots + (-)^{n-1} \frac{f_n(a) - f_n(a + \omega_1)}{n!} z^n + \ldots$$

But in the "Theory of the Gamma Function," § 20, we obtained the expansion which may be written

$$\frac{-ze^{-az}}{1 - e^{-\omega_2 z}} = -S_1^{(2)}(a \mid \omega_2) + \frac{S_1'(a \mid \omega_2)}{1!} z + \ldots + (-)^{n-1} \frac{S'_n(a \mid \omega_2)}{n!} z^n + \ldots$$

Equating coefficients in these two expansions, we obtain

$$A_1(a + \omega_1) - A_1(a) = 0$$

$$A_0(a + \omega_1) - A_0(a) = S_1^{(2)}(a \mid \omega_2)$$

$$f_1(a + \omega_1) - f_1(a) = S'_1(a \mid \omega_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_n(a + \omega_1) - f_n(a) = S'_n(a \mid \omega_2)$$

and it is obvious that a similar set of equations hold in which ω_1 and ω_2 are interchanged.

Hence $A_1(a)$ is a constant whose value, from the first term of the expansion, is

$$\frac{1}{\omega_1 \omega_2} = {}_2S_1^{(3)}(a \mid \omega_1, \omega_2) \quad \text{by § 4.}$$

Again $A_0(a) = {}_2S_1^{(2)}(a \mid \omega_1, \omega_2)$, for these two expressions can only differ by a constant which by § 4 and the actual expansion is at once seen to vanish.

Finally for all positive integral values of n, $f_n(a)$ can only differ from ${}_2S'_n(a \mid \omega_1, \omega_2)$ by a constant,—*i.e.*, let us say,

$$f_n(a) = {}_2S'_n(a \mid \omega_1, \omega_2) + \mu_n.$$

Differentiate the expansion thus obtained with respect to a, and we find

$$\frac{-z^2 e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S_1^{(3)}(a) - \frac{{}_2S_1^{(2)}(a)}{1!}z + \ldots + (-)^{n-1}\frac{{}_2S_n^{(2)}(a)}{n!}z^n + \ldots$$

and hence we have finally (§ 6)

$$\frac{ze^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2S_1^{(3)}(a)}{z} - {}_2S_1^{(2)}(a) + \frac{{}_2S'_1(a)}{1!}z + \ldots + (-)^{n-1}\frac{{}_2S'_n(a)}{n!}z^n + \ldots$$

which is the expansion required.

This expansion may be used to define the double Bernoullian numbers, and all their properties may be deduced from it. A procedure analogous to the one here suggested will be the one employed in the general theory of multiple Bernoullian functions.

§ 16. Several expansions of constant occurrence may be deduced from the one just obtained.

In the first place, note that we may write the expansion in the form

$$\frac{ze^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} - \frac{a}{\omega_1\omega_2} + \frac{{}_2S'_1(a)}{1!}z + \ldots$$
$$+ (-)^{n-1}\frac{{}_2S'_n(a)}{n!}z^n + \ldots$$

Put now $a = 0$, and we have by definition of the double Bernoullian numbers,

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} + {}_2B_1(\omega_1, \omega_2)z + \ldots$$
$$+ (-)^{n-1}\frac{{}_2B_n(\omega_1,\omega_2)}{(n-1)!}z^n + \ldots$$

We thus have (§ 11)

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} + \frac{{}_2B_1(\omega_1, \omega_2)}{0!}z + \frac{{}_2B_3(\omega_1, \omega_2)}{2!}z^3 + \ldots$$
$$+ \frac{{}_2B_{2n+1}(\omega_1, \omega_2)}{(2n)!}z^{2n+1} + \ldots$$
$$+ \sum_{m=1}^{\infty}(-)^{m-1}\frac{B_m(\omega_1^{2m-1}+\omega_2^{2m-1})z^{2m}}{2\,.\,(2m)!}.$$

And the last series may be written

$$\tfrac{1}{2}\left[\frac{z}{1-e^{-\omega_1 z}} + \frac{z}{1-e^{-\omega_2 z}} - \frac{1}{\omega_1} - \frac{1}{\omega_2} - z\right].$$

Hence we find

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{2}\frac{z}{(1-e^{-\omega_1 z})} - \frac{z}{2(1-e^{-\omega_2 z})} + \frac{z}{2}$$
$$= \frac{1}{\omega_1\omega_2 z} + \sum_{n=0}^{\infty} \frac{{}_2B_{2n+1}(\omega_1, \omega_2)}{(2n)!} z^{2n+1}$$

as the expansion from which the odd double Bernoullian numbers may be derived.

Finally if we integrate the fundamental expansion of § 15 with respect to a between 0 and a, we obtain

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2S_1^{(2)}(a) - {}_2S_1^{(2)}(o)}{z} - [{}_2S'_1(a) - {}_2S'_1(o)] + \frac{{}_2S_1(a)}{1!}z + \ldots$$
$$\ldots + \frac{(-)^{n-1}{}_2S_n(a)}{n!}z^n + \ldots$$

or, as we may write it,

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{a}{\omega_1\omega_2 z} - {}_2S_0(a) + \frac{{}_2S_1(a)}{1!}z + \ldots + \frac{(-)^{n-1}{}_2S_n(a)}{n!}z^n + \ldots$$

as the expansion from which the double Bernoullian functions themselves are at once obtained.

All these expansions are valid within the circle whose radius is the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|$ and $\left|\frac{2\pi\iota}{\omega_2}\right|$.

§ 17. Hitherto we have considered the double Bernoullian function as defined by one of two difference equations, each of which involves the simple Bernoullian function.

We proceed now to prove that, to a linear function of a, ${}_2S_n(a)$ is the only rational integral algebraic function of a satisfying the difference equation

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = a^n.$$

In the first place it is at once evident that ${}_2S_n(a)$ does satisfy this equation.

Again the difference of any two solutions is a solution of

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = 0.$$

Putting $$f(a+\omega_1) - f(a) = \phi(a),$$
we have $$\phi(a+\omega_2) - \phi(a) = 0.$$

Hence, if $f(a)$ is the difference of two algebraic solutions of the original equation, $\phi(a)$ will be an algebraic simply periodic function, and therefore a constant.

And thus we shall have

$$f(a+\omega_1) - f(a) = \text{constant},$$

so that if $f(a)$ is to be an algebraic polynomial, it must be of the form

$$\lambda a + \mu,$$

where λ and μ are constant with respect to a.

Thus the difference of any two rational integral algebraic solutions of

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f(a + \omega_2) + f(a) = a^n$$

is of the form $\lambda a + \mu$. Whence the theorem in question.

[Dr. E. W. HOBSON has kindly pointed out to me that the analysis of the preceding paragraphs would be much simplified by starting from the direct definition of the double Bernoullian function in § 17.

We should thus define the double Bernoullian function by the expansion

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{a}{\omega_1\omega_2 z} - {}_2S_0(a) + \ldots + (-)^{n-1}z^n \frac{{}_2S_n(a|\omega_1, \omega_2)}{n!} + \ldots$$

On differentiating with respect to a, we get the expression of § 15.

From the relation

$$\frac{1-e^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{e^{-az}}{1-e^{-\omega_2 z}},$$

we obtain the fundamental difference relations for the double Bernoullian function.

The result of § 10 follows from the identity

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1-e^{(\omega_1+\omega_2-a)z}}{(1-e^{\omega_1 z})(1-e^{\omega_2 z})} = \frac{1}{1-e^{-\omega_1 z}} - \frac{1}{(1-e^{-\omega_2 z})} - 1,$$

and that of § 14 from

$$\sum_{k=0}^{p-1}\sum_{l=0}^{q-1}\left\{1 - e^{-\left(a+\frac{k\omega_1}{p}+\frac{l\omega_2}{q}\right)z}\right\} = pq - \frac{e^{-az}(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}{(1-e^{-\frac{\omega_1 z}{p}})(1-e^{-\frac{\omega_2 z}{q}})}.$$

Inasmuch as theoretically the properties of an algebraical polynomial should not be derived from consideration of the coefficients of an infinite series, the original investigation has been retained. I had already proposed to myself to work out the theory of multiple Benoullian functions by a method closely allied to that suggested by Dr. HOBSON.—*Note added July* 3, 1900.]

PART II.

The Double Gamma Function $\Gamma_2(a|\omega_1,\omega_2)$ *and its Elementary Properties.*

§ 18. In the elementary consideration of the simple gamma function it was found to be necessary to rely on two algebraical limit theorems :—

(1) EULER'S theorem $\underset{n=\infty}{\mathrm{L}t}\left[1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n\right] = \gamma.$

(2) STIRLING'S theorem $\underset{n=\infty}{\mathrm{L}t}\left[\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}\right] = \sqrt{2\pi}.$

In an analogous treatment of the double gamma function we may expect that similar limit theorems will be required. This, in fact, is the case; but for our present purpose it is sufficient to take particular cases of the asymptotic expansion for $\log \Gamma(z)$.

To make use of this approximation we need only remember that ("Theory of the Gamma Function," § 39) if z and ω be any finite complex quantities, and n a positive integer,

$$\log \prod_{m=0}^{n} (z + m\omega) = \log \Gamma_1 [z + (n+1)\omega \,|\, \omega] - \log \Gamma_1 (z \,|\, \omega).$$

We suppose that such values of the logarithms are chosen that additive terms involving $2\pi\iota$ do not enter. In other words, we shall say that the logarithms have their absolute values, the formula just written being merely a convenient way of writing the identity

$$\prod_{\omega=0}^{n} (z + m\omega) = \frac{\Gamma_1[z + (n+1)\omega]}{\Gamma_1(z)}.$$

On differentiating this identity with respect to z we have

$$\sum_{\omega=0}^{n} \frac{1}{(z + m\omega)} = \psi_1^{(1)}[z + (n+1)\omega \,|\, \omega] - \psi_1^{(1)}(z \,|\, \omega)$$

with the notation of § 2 of the "Theory of the Gamma Function."

§ 19. The double gamma function of z with parameters ω_1 and ω_2 we write

$$\Gamma_2 (z \,|\, \omega_1, \omega_2).$$

When there is no doubt as to their presence the parameters are omitted. From this function we form the subsidiary system

$$\psi_2^{(1)} (z \,|\, \omega_1, \omega_2) = \frac{d}{dz} \log \Gamma_2 (z \,|\, \omega_1, \omega_2).$$

$$\psi_2^{(2)} (z \,|\, \omega_1, \omega_2) = \frac{d^2}{dz^2} \log \Gamma_2 (z \,|\, \omega_1, \omega_2).$$

and so on.

As a definition we assume

$$\psi_2^{(3)}(z \,|\, \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3}.$$

This double series is, by EISENSTEIN'S Theorem,* absolutely convergent, provided the ratio of ω_1 to ω_2 is not real and negative. This limitation on the parameters holds throughout the whole theory of the double gamma functions. It corresponds to the limitation in WEIERSTRASS' theory of elliptic functions that Γ must not be a real quantity.

* *v.* FORSYTH, "Theory of Functions," § 56.

We shall show that by successive integrations we may determine $\Gamma_2(z \mid \omega_1, \omega_2)$ as a function symmetrical in ω_1 and ω_2 such that

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota S'_1(z \mid \omega_2)},$$

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\rho_1(\omega_1)} e^{-2m'\pi\iota S'_1(z \mid \omega_1)},$$

where $\rho_1(\omega) = \sqrt{(2\pi/\omega)}$ ("Theory of the Gamma Function," § 31), and m and m' are integers (unity or zero), to be determined in accordance with the detailed theory which we proceed to give.

And the function so determined will be unique, provided

$$\underset{z=0}{Lt}[z\Gamma_2(z \mid \omega_1, \omega_2)] = 1.$$

§ 20. We readily see that the function

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z+\Omega)^3},$$

where $\Omega = m_1\omega_1 + m_2\omega_2$ satisfies the two difference equations

$$\psi_2^{(3)}(z+\omega_1) = \psi_2^{(3)}(z) - \psi_1^{(3)}(z \mid \omega_2)$$
$$\psi_2^{(3)}(z+\omega_2) = \psi_2^{(3)}(z) - \psi_1^{(3)}(z \mid \omega_1),$$

where here, as always, we suppress the parameters of the functions $\psi_2^{(r)}(z)$ $(r = 1, 2, \ldots)$ and $\Gamma_2(z)$ when these parameters are supposed to exist in perfectly general form.

For we have at once from the definition-series

$$\psi_2^{(3)}(z+\omega_1) - \psi_2^{(3)}(z) = -2\left[-\sum_{m_2=0}^{\infty} \frac{1}{(z+m_2\omega_2)^3}\right]$$
$$= -\psi_1^{(3)}(z \mid \omega_2). \qquad \text{("Gamma Function," § 2)}$$

Next, we may show that the function

$$\psi_{21}^{(2)}(z \mid \omega_1\ \omega_2) = -\gamma_{21}(\omega_1, \omega_2) + \frac{1}{z^2} + \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \left[\frac{1}{(z+\Omega)^2} - \frac{1}{\Omega^2}\right]$$

satisfies the two difference equations

$$\psi_2^{(2)}(z+\omega_1) = \psi_2^{(2)}(z) - \psi_1^{(2)}(z \mid \omega_2)$$
$$\psi_2^{(3)}(z+\omega_2) = \psi_2^{(2)}(z) - \psi_1^{(2)}(z \mid \omega_1),$$

whatever be the value of the constant $\gamma_{21}(\omega_1, \omega_2)$.

For the series for $\phi_2^{(2)}(z)$ is absolutely convergent so long as

$$\frac{1}{(z+\Omega)^2} - \frac{1}{\Omega^2}$$

is regarded as one term, and we may subtract two absolutely convergent series by a term-by-term process. Hence we have immediately

$$\psi_2^{(2)}(z+\omega_1) - \psi_2^{(2)}(z) = -\sum_{m_2=0}^{\infty} \frac{1}{(z+m_2\omega_2)^2},$$

and therefore (" Theory of the Gamma Function," § 2)

$$\psi_2^{(2)}(z+\omega_1) - \psi_2^{(2)}(z) = -\psi_1^{(2)}(z \mid \omega_2).$$

Similarly,
$$\psi_2^{(2)}(z+\omega_2) - \psi_2^{(2)}(z) = -\psi_1^{(2)}(z \mid \omega_1).$$

§ 21. It may now be shown that the function

$$-\psi_2^{(1)}(z \mid \omega_1, \omega_2) = z\gamma_{21}(\omega_1, \omega_2) + \gamma_{22}(\omega_1, \omega_2) + \tfrac{1}{2} + \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}{}' \left[\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right]$$

satisfies the two difference relations

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z \mid \omega_2) + \frac{2m\pi\iota}{\omega_2},$$

$$\psi_2^{(1)}(z+\omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z \mid \omega_1) + \frac{2m'\pi\iota}{\omega_1},$$

for certain values of the numbers m and m', provided

$$\gamma_{21}(\omega_1, \omega_2) = -\operatorname*{Lt}_{n=\infty}\left[\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} - \frac{1}{\omega_1\omega_2}\log n + \frac{1}{\omega_1\omega_2}\left\{\log(\omega_1+\omega_2) - \log\omega_1 - \log\omega_2\right\}\right],$$

the principal values of the logarithms being taken.

We may write $\psi_2{}^{(1)}(z)$ in the form

$$-\operatorname*{Lt}_{n=\infty}\left[z\gamma_{21}(\omega_1, \omega_2) + \gamma_{22}(\omega_1, \omega_2) + \frac{1}{z} + \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \left\{\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right\}\right],$$

and now we obtain at once

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z)$$
$$= -\omega_1\gamma_{21}(\omega_1, \omega_2) + \operatorname*{Lt}_{n=\infty}\left[\sum_{m_2=0}^{n}\frac{1}{z+m_2\omega_2} - \sum_{m_2=0}^{n}\frac{1}{z+(n+1)\omega_1+m_2\omega_2} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega_2}\right].$$

Hence we may take

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z \mid \omega_2) + 2\frac{m\pi\iota}{\omega_2},$$

provided

$$\omega_1\gamma_{21}(\omega_1, \omega_2) = \psi_1^{(1)}(z \mid \omega_2) - 2\frac{m\pi\iota}{\omega_2}$$
$$+ \operatorname*{Lt}_{n=\infty}\left[\sum_{m_2=}^{n}\frac{1}{z+m_2\omega_2} - \sum_{m_2=0}^{n}\frac{1}{z+m_2\omega_2+(n+1)\omega_1} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2}\right].$$

or, utilising § 19 Corollary, provided

$$\omega_1\gamma_{21}(\omega_1, \omega_2) + 2\frac{m\pi i}{\omega_2} = \underset{n=\infty}{\mathrm{Lt}}\Big[\psi_1^{(1)}[z + (n+1)\,\omega_2\,|\,\omega_2] + \psi_1^{(1)}[z + (n+1)\,\omega_1\,|\,\omega_2]$$

$$- \psi_1^{(1)}[z + (n+1)\,.\,(\omega_1 + \omega_2)\,|\,\omega_2] - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2}\Big].$$

But ("Gamma Function," Part IV.) we know that when $|z|$ is very large, and $\frac{z}{\omega}$ not real and negative,

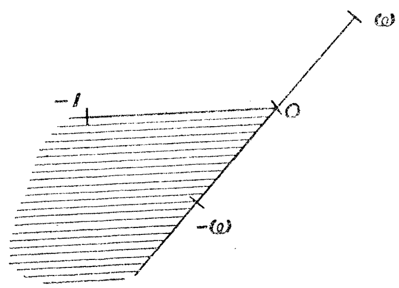
$$\log \Gamma_1(z + a\,|\,\omega) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\left\{\log\frac{z}{\omega} + \log\omega\right\} - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$

+ terms which vanish when $|z|$ becomes infinite.

In every case the principal value of the logarithm is to be taken, *i.e.*, that value whose amplitude lies between $-\pi$ and π.

Now
$$\log\frac{z}{\omega} + \log\omega = \log z$$

in all cases except when z lies in the region formed by lines from the origin to the points $-\omega$ and -1 (shaded in the figure).



When z does lie within this region, we readily see that

$$\log\frac{z}{\omega} + \log\omega = \log z + 2\pi\iota$$

if I (ω), the imaginary part of ω, is positive, and

$$\log\frac{z}{\omega} + \log\omega = \log z - 2\pi\iota$$

if I (ω), the imaginary part of ω, is negative.

Thus

$$\log \Gamma_1(z + a\,|\,\omega) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\{\log z + 2k\pi\iota\} - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{z\pi}{\omega}$$

+ terms which vanish when $|z|$ is infinite,

where $k = 0$, unless z lie within the region between the axes to -1 and $-\omega$, and where $k = \pm 1$, the upper or lower sign being taken according as $I(\omega)$ is positive or negative, when z does lie within this region.

On differentiating, we have the derived expansion*

$$\psi_1^{(1)}(z + a \mid \omega) = \frac{1}{\omega}(\log z + 2k\pi\iota)$$
$$+ \text{ terms which vanish when } |z| \text{ becomes infinite,}$$

the principal value of the logarithm being again taken, and k being determined as before.

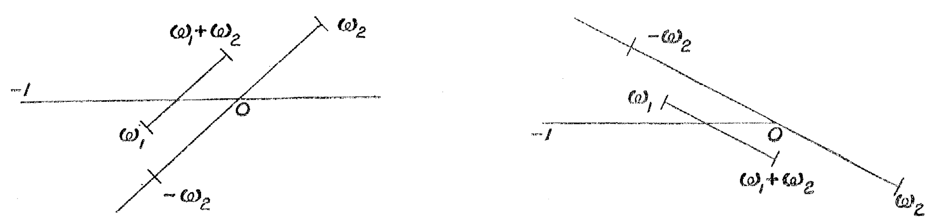
Substituting in the expression for $\gamma_{21}(\omega_1, \omega_2)$, we have

$$\omega_1\gamma_{21}(\omega_1, \omega_2) + 2\frac{m\pi i}{\omega_2} = \underset{n=\infty}{\mathrm{Lt}}\left[\frac{1}{\omega_2}\log n\omega_2 + \frac{1}{\omega_2}\log n\omega_1 \right.$$
$$\left. - \frac{1}{\omega_2}\log n \,.\, (\omega_1 + \omega_2) - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2} + \frac{2k\pi\iota}{\omega_2}\right],$$

where $k = 0$, unless $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ or $\left.\begin{matrix}(\omega_1 + \omega_2) \text{ does} \\ \omega_1 \text{ does not}\end{matrix}\right\}$

lie in the region bounded by lines from the origin to -1 and $-\omega_2$. [It is understood, of course, that the principal values of the logarithms are to be taken.]

When, as in the figures $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region of exception, $k = \pm 1$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.



From the diagrams, we see at once that it is impossible that $\left.\begin{matrix}(\omega_1 + \omega_2) \text{ should} \\ \omega_1 \text{ should not}\end{matrix}\right\}$ lie within the region bounded by the lines from the origin to -1 and ω_2.

Take now $m = k$, that is to say, let m be such that we have $m = 0$, unless $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie in the region of exception, and $m = \pm 1$ according as $I(\omega_2)$ is positive or negative, when this exceptional circumstance takes place.

* According to M. Poincaré, we may not in general differentiate an asymptotic expression. The one in question, however, may be readily established by the methods employed for $\log \Gamma_1(z + a \mid \omega)$.

Then

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\frac{1}{\omega_1\omega_2}\{\log n\omega_2 + \log n\omega_1 - \log n(\omega_1+\omega_2)\} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}\frac{1}{\Omega^2}\right].$$

But this expression is symmetrical in ω_1 and ω_2; and we must therefore have the analogous relation

$$\psi_2^{(1)}(z+\omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + \frac{2m'\pi\iota}{\omega_1},$$

where $m' = 0$, unless $\left.\begin{matrix}\omega_2 \text{ does} \\ (\omega_1+\omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes from the origin to $-\omega_1$ and -1, in which case $m' = \pm 1$, the upper or lower sign being taken as I (ω_1) is positive or negative.

Provided therefore that

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\frac{1}{\omega_1\omega_2}\log n - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}'\frac{1}{\Omega^2} + \frac{1}{\omega_1\omega_2}\{\log\omega_1 + \log\omega_2 - \log(\omega_1+\omega_2)\}\right],$$

we have, with the assigned values of m and m', the two difference relations

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + \frac{2m\pi\iota}{\omega_2},$$

$$\psi_2^{(1)}(z+\omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_1) + \frac{2m'\pi\iota}{\omega_1}.$$

The function γ_{21}, (ω_1, ω_2) we propose to call the first double gamma modular form. It will subsequently be expressed in terms of the function D(τ) introduced into the theory of the functions G $(z\,|\,\tau)$ ("Genesis of the Double Gamma Function," § 4).

It will be seen later that the algebra of the double gamma function would have been slightly simplified had a modified value been taken for this function $\gamma_{21}(\omega_1, \omega_2)$, and the analogue shortly to be considered, $\gamma_{22}(\omega_1, \omega_2)$. I did not observe this fact until the theory had been completely developed, and the matter is scarcely of sufficient importance to demand the labour which such a change would entail.

Corollary.—Notice that it has been proved incidentally that

$$\sum_{m_1=0}^{n}\sum_{m=-\infty}^{n}{}'\frac{1}{(m_1\omega_1 + m_2\omega_2)^2}$$

is infinite, when n is infinite, like $\dfrac{1}{\omega_1\omega_2}\log n$.

§ 22. As the numbers in m and m' enter constantly into the analysis, it is necessary to consider their properties.

Suppose that the functions $\log z$, $\log_{\omega_1} z$, $\log_{\omega_2} z$ are natural logarithms (with e as base), which are real when z is real and positive, and which are rendered uniform by cross-cuts along the axes of -1, $-\omega_1$ and $-\omega_2$ respectively.

Then it is readily seen that

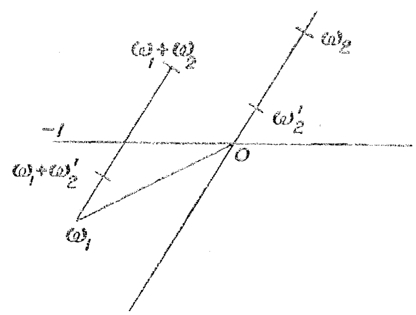
$$\log_{\omega_1}(\omega_1+\omega_2)-\log_{\omega_1}\omega_1=\log(\omega_1+\omega_2)-\log\omega_1-2m\pi\iota$$
$$\log_{\omega_1}(\omega_1+\omega_2)-\log_{\omega_1}\omega_2=\log(\omega_1+\omega_2)-\log\omega_2-2m'\pi\iota$$
$$\log_{\omega_2}(\omega_1+\omega_2)-\log_{\omega_2}\omega_2=\log(\omega_1+\omega_2)-\log\omega_2-2m'\pi\iota$$
$$\log_{\omega_2}(\omega_1+\omega_2)-\log_{\omega_2}\omega_1=\log(\omega_1+\omega_2)-\log\omega_1-2m\pi\iota.$$

By inspection of a diagram we see that m and m' both vanish if the difference of the amplitudes of ω_1 and ω_2 is less than π, these amplitudes being measured between 0 and $\pm\pi$ positively or negatively from the positive half of the real axis. In particular when the real parts of ω_1 and ω_2 are both positive, m and m' both vanish. Not only so, but in all cases either m or m' must vanish.

Again, if the difference of the amplitudes of ω_1 and ω_2 is greater than π, m and m cannot both vanish. In fact, in this case we have the important relation

$$m-m'=\pm 1,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative or positive. This result is intuitive geometrically; in the figure, for instance, two cases are indicated in which $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative.



For corresponding to the unaccented value of ω_2,

$$\left.\begin{aligned} m &= 1 \\ m' &= 0 \end{aligned}\right\},$$

and corresponding to the accented value of ω_2,

$$\left.\begin{aligned} m &= 0 \\ m' &= -1 \end{aligned}\right\}.$$

Thus in both cases $m-m'=1$.

No such simple expression can be given for $m+m'$, a number which is of constant occurrence in the higher theory.

However, from the values for m and m' previously given, we see that when the axes of ω_1 and ω_2 include the axis of -1 within an angle less than two right angles, the values of $(m+m')$ are given by the table

$(m + m')$	$I(\omega_1)$	$I(\omega_1 + \omega_1)$
1	+ ve	+ ve
−1	+ ve	− ve
−1	− ve	+ ve
1	− ve	− ve

and therefore $m + m' = \pm 1$, the upper or lower signs being taken according as $I(\omega_1 + \omega_2)$ and $I(\omega_1)$ have the same or opposite signs.

§ 23. It may now be shown that, if C have any arbitrary value, the function

$$\Gamma_2^{-1}(z \mid \omega_1, \omega_2) = C\, e^{\frac{z^2}{2}\gamma_{21}(\omega_1, \omega_2) + z\gamma_{22}(\omega_1, \omega_2)} \,.\, z \times \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$, will satisfy the two difference relations

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\sqrt{\frac{2\pi}{\omega_2}}}\, e^{-2m\pi\iota\left(\frac{z}{\omega_2} - \frac{1}{2}\right)};$$

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\sqrt{\frac{2\pi}{\omega_1}}}\, e^{-2m'\pi\iota\left(\frac{z}{\omega_1} - \frac{1}{2}\right)};$$

where m and m' are the numbers previously specified, and

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_0^n \sum_0^n{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - \frac{n+1}{\omega_2}\log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n+1}{\omega_1}\log\left(1 + \frac{\omega_1}{\omega_2}\right) + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\right],$$

the principal values of the logarithms being taken.

Observe that with the notation previously introduced ("Theory of the Gamma Function," §§ 16 and 31) we may write these difference relations in the form

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\rho_1(\omega_2)}\, e^{-2m\pi\iota\, S'_1(z \mid \omega_2)}$$

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\rho_1(\omega_1)}\, e^{-2m'\pi\iota\, S'_1(z \mid \omega_1)}.$$

The proof, to which we now proceed, is exactly analogous to the one just given.

We have

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = e^{\frac{\gamma_{21}}{2}\omega_1^2 + \gamma_{21}z\omega_1 + \gamma_{22}\omega_1} \cdot \frac{z + \omega_1}{z} \cdot \underset{n=\infty}{\mathrm{Lt}} \left[\prod_{m_1=0}^{n} \prod_{m_2=0}^{n}{}' \left\{\frac{z + (m_1 + 1)\omega_1 + m_2\omega_2}{z + m_1\omega_1 + m_2\omega_2} \times e^{-\frac{\omega_1}{\Omega} + \frac{2z\omega_1 + \omega_1^2}{2\Omega^2}}\right\}\right].$$

where γ_{21} and γ_{22} are understood, as always, to mean $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$.

Substitute now the value of $\gamma_{21}(\omega_1, \omega_2)$ obtained in § 21, and we find

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \exp.\left[\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\{\log\omega_2+\log\omega_1-\log(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right].\frac{z+\omega_1}{z}.$$

$$\times \underset{n=\infty}{\mathrm{L}t}\left[\exp.\left\{\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\log n\right\}\prod_{m_1=0}^{n}\prod_{m_2=0}^{n}{}'\left\{\frac{z+(m_1+1)+m_2\omega_2}{z+m_1\omega_1+m_2\omega_2}e^{-\frac{\omega_1}{\Omega}}\right\}\right],$$

$$= \exp.\left[\frac{2z\omega_1+\omega_2}{2\omega_1\omega_2}\{\log\omega_1+\log\omega_2-\log(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right].\Gamma_1(z\,|\,\omega_2)$$

$$\times \underset{n=\infty}{\mathrm{L}t}\left[\exp.\left\{\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\log n-\sum_0^n\sum_0^n{}'\frac{\omega_1}{\Omega}\right\}.\frac{\Gamma_1[z+(n+1).(\omega_1+\omega_2)\,|\,\omega_2]}{\Gamma_1[z+(n+1)\,\omega_1\,|\,\omega_2]\,\Gamma_1[z+(n+1)\omega_2\,|\,\omega_2]}\right]$$

by the employment of the identity of § 18. Their principal values must throughout be assigned to the logarithms.

But, as has been seen in § 21, from the formula obtained in the "Theory of the Gamma Function," § 41, we have when $|z|$ is large and z not in the vicinity of the axis of $-\omega$,

$$\log\Gamma_1(z+a\,|\,\omega) = \left(\frac{z+a}{\omega}-\tfrac{1}{2}\right)\{\log z+2k\pi\iota\}-\frac{z}{\omega}+\log\rho_1(\omega)$$
$$+\text{ terms which vanish when } |z| \text{ is infinite,}$$

where $k = 0$, unless z lies within the region between the axes of -1 and $-\omega$, in which case $k = \pm 1$, the upper or lower sign being taken as $\mathrm{I}(\omega)$ is positive or negative. The principal value of $\log z$ is to be taken, and the prescription to be given to $\log\Gamma_1(z+a\,|\,\omega)$ is left indeterminate: it is obvious that we only get additive terms involving $2\pi\iota$, which vanish in the sequel.

Inasmuch as when n is large, none of the points

$$z+(n+1)(\omega_1+\omega_2),\quad z+(n+1)\,\omega_1,\ \text{and}\ z+(n+1)\,\omega_2$$

lie in the vicinity of the negative direction of the axis of ω_2, we may substitute the values given by the asymptotic expansion in the expression for $\dfrac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)}$.

We shall find

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\,\Gamma_1(z\,|\,\omega_2)} = \exp.\ \underset{n=\infty}{\mathrm{L}t}\left[\frac{2z\omega_1+\omega_1^2}{z\omega_1\omega_2}\{\log n\omega_2+\log\omega_1-\log(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right.$$
$$-\sum_0^n\sum_0^n{}'\frac{\omega_1}{\Omega}+\left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2}-\tfrac{1}{2}\right)\log n(\omega_1+\omega_2)-\frac{n(\omega_1+\omega_2)}{\omega_2}$$
$$-\log\rho_1(\omega_2)-\left(\frac{z+(n+1)\,\omega_2}{\omega_2}-\tfrac{1}{2}\right)\log n\omega_2$$
$$-\left(\frac{z+(n+1)\,\omega_1}{\omega_2}-\tfrac{1}{2}\right)\log n\omega_1+\frac{n(\omega_1+\omega_2)}{\omega_2}$$
$$\left.+2\left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2}-\tfrac{1}{2}\right)k_1\pi\iota-2\left(\frac{z+(n+1)\,\omega_1}{\omega_2}\right)-\tfrac{1}{2}k_2\pi\iota\right]$$

In this expression their principal values are throughout to be assigned to the logarithms, and the numbers k are to be such that

$k_1 = 0$, unless $(\omega_1 + \omega_2)$ has within the region bounded by axes to -1 and $-\omega_2$, in which case

$k_1 = \pm 1$, the upper or lower sign being taken as $I(\omega_2)$ is positive or negative, while

$k_2 = 0$, unless ω_1 lies within the region bounded by axes to -1 and $-\omega_2$, in which case

$k_2 = \pm 1$, the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

On reduction we now see that

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\Gamma_1(z\,|\,\omega_2)} = \exp. \underset{n=\infty}{\mathrm{L}t}\Big[\omega_1\gamma_{22}(\omega_1, \omega_2) + \sum_0^n \sum_0^{n}{}' \frac{\omega_1}{\Omega} + (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1}\log n(\omega_1+\omega_2)$$
$$- \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_2}\log n\omega_2 - \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 - \log\rho_1(\omega_2)$$
$$+\left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2} - \tfrac{1}{2}\right)2k_1\pi i - \left(\frac{z+(n+1)\,\omega_1}{\omega_2} - \tfrac{1}{2}\right)2k_2\pi\iota\Big].$$

We must consider the three possible cases in which k_1 and k_2 do not both vanish.

(1) Firstly, when $\left\{\begin{matrix}\omega_1 \text{ does} \\ (\omega_1+\omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_2$.

In this case $\left\{\begin{matrix}k_1 = 0 \\ k_2 = \pm 1\end{matrix}\right\}$ the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

And we have

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\Gamma_1(z\,|\,\omega_2)}\sqrt{\frac{2\pi}{\omega_2}} = \underset{n=\infty}{\mathrm{L}t}\exp.\Big[\omega_1\Big\{\gamma_{22} - \sum_0^n\sum_0^{n}{}'\frac{1}{\Omega} + (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2)$$
$$- \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2 - \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1$$
$$\pm 2\frac{(n+1)\,\pi\iota}{\omega_2}\Big\} \mp 2\left(\frac{z}{\omega_2} - \tfrac{1}{2}\right)\pi\iota\Big].$$

the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

But in this case $m = \pm 1$, the signs being chosen in the same way.

If then we take

$$\gamma_{22}(\omega_1, \omega_2) = \sum_0^n\sum_0^{n}{}'\frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2) + \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2$$
$$+ \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \pm 2\frac{(n+1)\,\pi\iota}{\omega_2}$$

we shall have

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\mid\omega_2)}{\sqrt{(2\pi/\omega_2)}} e^{-2m\left(\frac{z}{\omega_2}-\frac{1}{2}\right)\pi\iota} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

(2) Secondly, when $\begin{Bmatrix} \omega_1 \text{ does} \\ (\omega_1+\omega_2) \text{ does} \end{Bmatrix}$ lie within the region bounded by axes to -1 and $-\omega_2$.

In this case $\begin{matrix} k_1 = \pm 1, \\ k_2 = \pm 1, \end{matrix} \quad m = 0.$

We shall have then the same relation

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z)\omega_2}{\sqrt{(2\pi/\omega_2)}} e^{-2m'\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)},$$

provided we take

$$\gamma_{22}(\omega_1, \omega_2) = \sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2} \log n(\omega_1+\omega_2) + \frac{(2n+1)\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2$$
$$+ \frac{(2n+1)\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \mp 2\frac{(n+1)\pi\iota}{\omega_1},$$

the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

(3) The third case, when $\begin{Bmatrix} \omega_1 \text{ does not} \\ (\omega_1+\omega_2) \text{ does} \end{Bmatrix}$ lie within the region bounded by axes -1 and $-\omega_2$, is easily seen to be impossible.

In all other cases we shall have the relation (1), provided

$$\gamma_{22}(\omega_1, \omega_2) = \sum_0^n \sum_0^{n_0}{}' \frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2} \log n(\omega_1+\omega_2) + \frac{(2n+1)\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2$$
$$+ \frac{(2n+1)\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \quad . \quad . \quad . \quad . \quad (2).$$

Suppose now that we had investigated similarly the quotient

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)},$$

we should have obtained the difference equation

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\mid\omega_1)}{\sqrt{(2\pi/\omega_1)}} e^{-2m'\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)},$$

where $\gamma_{21}(\omega_1, \omega_2)$ has the value D, let us say, given by equation (2), except in two cases.

(1)′ When $\begin{Bmatrix} \omega_2 \text{ does} \\ (\omega_1+\omega_2) \text{ does not} \end{Bmatrix}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

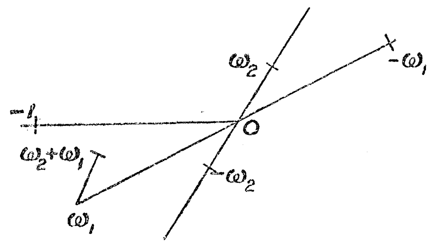
$$\gamma_{22}(\omega_1, \omega_2) = D \pm 2\frac{(n+1)\pi\iota}{\omega_1},$$

the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative.

(2)′ When $\left\{\begin{matrix}\omega_2 \text{ does} \\ (\omega_1+\omega_2) \text{ does}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \mathrm{D} \mp 2\frac{(n+1)\pi\iota}{\omega_2}.$$

But the cases (1) and (2)′ are precisely the same, as an inspection of the figure shows at once, except that $\mathrm{I}(\omega_2)$ is positive when $\mathrm{I}(\omega_1)$ is negative, and *vice versâ.*



And, similarly, the cases (2) and (1)′ are the same, with a similar change.

Hence the values which must be assigned to $\gamma_{22}(\omega_1, \omega_2)$, in order that the equations

$$\left\{\begin{aligned} \frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} &= \frac{\Gamma_1(z|\omega_2)}{\sqrt{2\pi/\omega_2}}\, e^{-2m\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)} \\ \frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} &= \frac{\Gamma_1(z|\omega_1)}{\sqrt{2\pi/\omega_1}}\, e^{-2m'\pi\iota\left(\frac{z}{\omega_1}-\frac{1}{2}\right)} \end{aligned}\right.$$

may co-exist, are precisely the same.

We shall have then these two equations, provided

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\, \mathrm{D},$$

where D stands for
$$\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \left(n+\tfrac{1}{2}\right)\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log\left[n(\omega_1+\omega_2)\right]$$
$$+\frac{(2n+1)\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2 + \frac{(2n+1)\omega_1-\omega_2}{2\omega_2\omega_1}\log n\omega_1$$

(the principal values of the logarithms being taken), except in two cases,

(1) When $\left\{\begin{matrix}\omega_1 \text{ does} \\ (\omega_1+\omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_2$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\mathrm{D} + 2\frac{m(n+1)\pi\iota}{\omega_2}\right].$$

(2) When $\left\{\begin{matrix}\omega_2 \text{ does} \\ (\omega_1+\omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\mathrm{D} + 2\frac{m'(n+1)\pi\iota}{\omega_1}\right].$$

Since m or m' always vanishes, we see, on combining these results, that in all cases

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\text{L}t}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - (n + \tfrac{1}{2})\frac{\omega_1 + \omega_2}{\omega_1\omega_2}\log n(\omega_1 + \omega_2) + \frac{(2n+1)\,\omega_2 - \omega_1}{2\omega_1\omega_2}\log n\omega_2 + \frac{(2n+1)\,\omega_1 - \omega_2}{2\omega_1\omega_2}\log n\omega_1 + 2\left(\frac{m}{\omega_2} + \frac{m'}{\omega_1}\right)(n+1)\pi\iota\right]$$

$$= \underset{n=\infty}{\text{L}t}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1, \omega_2}\log n - (n+1)\frac{1}{\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - 2m\pi\iota\} - (n+1)\frac{1}{\omega_1}\{\log(\omega_1 + \omega_2) - \log\omega_2 - 2m'\pi\iota\} + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\right].$$

But

$$\log(\omega_1 + \omega_2) - \log\omega_1 - 2m\pi\iota = \log\left(1 + \frac{\omega_2}{\omega_1}\right)$$

$$\log(\omega_1 + \omega_2) - \log\omega_2 - 2m'\pi\iota = \log\left(1 + \frac{\omega_1}{\omega_1}\right),$$

the principal values of the logarithms being always taken. Hence

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\text{Lt}}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - \frac{n+1}{\omega_2}\log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n+1}{\omega_1}\log\left(1 + \frac{\omega_1}{\omega_2}\right) + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\right].$$

As a *corollary*, we see that, when n is very large,

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{m_1\omega_1 + m_2\omega_2} \text{ is infinite like } \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - \frac{n}{\omega_2}\log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n}{\omega_1}\log\left(1 + \frac{\omega_1}{\omega_2}\right).$$

§ 24. We now determine the constant C in the expression

$$\Gamma_2^{-1}(z \mid \omega_1, \omega_2) = \text{C}e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}} \,.\, z \,.\, \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right)e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right],$$

by the condition assigned in § 19, that

$$\underset{z=0}{\text{L}t}\, z\Gamma_2(z \mid \omega_1, \omega_2) = 1.$$

This at once gives us $\text{C} = 1$.

It is evident that, with the conditions of § 19, one and only one function can be constructed, and this is the double gamma function $\Gamma_2(z \mid \omega_1, \omega_2)$, which is such that

$$\Gamma_2^{-1}(z \mid \omega_1, \omega_2) = e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}} \cdot z \cdot \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right],$$

where $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$ are two constants, which we call the first and second double gamma modular functions of the parameters ω_1 and ω_2.

These constants are given by the relations

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\frac{1}{\omega_1\omega_2} \log n - \sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} + \frac{1}{\omega_1\omega_2}\{\log \omega_1 + \log \omega_2 - \log(\omega_1 + \omega_2)\}\right]$$

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \log n - \frac{n+1}{\omega_2} \log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n+1}{\omega_2} \log\left(1 + \frac{\omega_1}{\omega_2}\right) + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log \omega_1 - \log \omega_2\}\right],$$

where the logarithms are such that their principal values must always be taken.

And the theory is the natural extension of that of the simple gamma function $\Gamma_1(z \mid \omega_1)$, which is such that

$$\Gamma_1^{-1}(z \mid \omega_1) = e^{-\gamma_{11} z} \cdot z \cdot \prod_{m_1=1}^{\infty} \left[\left(1 + \frac{z}{m_1\omega_1}\right) e^{-\frac{z}{m_1\omega_1}}\right],$$

where the constant γ_{11} is given by the relation

$$\gamma_{11}(\omega_1) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_{m_1=1}^{n} \frac{1}{m_1\omega_1} - \frac{1}{\omega_1} \log n\omega_1\right],$$

and the principal value of the logarithm must again be taken.

§ 25. We may now see at once that

$$\Gamma_2(\omega_1 \mid \omega_1, \omega_2) = \sqrt{(2\pi/\omega_2)} \cdot e^{-m\pi\iota},$$

$$\Gamma_2(\omega_2 \mid \omega_1, \omega_2) = \sqrt{(2\pi/\omega_1)} \cdot e^{-m'\pi\iota}.$$

For we have seen that

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\sqrt{(2\pi/\omega_2)}} e^{-2m\pi\iota\left(\frac{z}{\omega_2} - \frac{1}{2}\right)},$$

where m is either zero or unity according to the determination of § 22.

But
$$\underset{z=0}{\mathrm{Lt}}\, [z\Gamma_2(z)] = 1,$$

and
$$\underset{z=0}{\mathrm{Lt}}\, [z\Gamma_1(z \mid \omega_2)] = 1,$$

as we see immediately from the product expression for

$$\Gamma_1(z \mid \omega_2).$$

Hence, making $z = 0$, we have

$$\Gamma_2^{-1}(\omega_1) = \frac{1}{\sqrt{(2\pi/\omega_2)}} e^{m\pi\iota},$$

which is one of the relations required.

We thus see that $\Gamma_2(\omega_1)$ is independent of ω_1, and not only so, but its value is substantially a quantity that appeared several times in the theory of the simple gamma function.

Thus we saw [§ 4 cor. "Gamma Function"] that

$$\prod_{q=1}^{n-1} \Gamma_1\left(\frac{q\omega}{n}\middle|\omega\right) = n^{-\frac{1}{2}}\{\sqrt{(2\pi/\omega)}\}^{n-1};$$

that (§ 8)

$$\int_0^\omega \log \Gamma_1(z\,|\,\omega)\,dz = \omega \log \sqrt{(2\pi/\omega)};$$

and the most general form of Stirling's theorem was seen to be

$$\begin{aligned}\log \prod_{m_1=0}^{pn} (a + m_1\omega) &= p\{n \log n - n\} + n\{S_1^{(2)}(a+\omega) p\omega \log p\omega\} \\ &+ \{1 + S_1'(a)\} \log n - \log \Gamma_1(a) + \log \sqrt{(2\pi/\omega)} \\ &+ S_1'(a+\omega) \log p\omega + \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} \frac{Sm(a+\omega) + {}_1B_{m+1}}{(p\omega)^m}.\end{aligned}$$

We now see an additional reason why it was proposed to write

$$\rho_1(\omega) = \sqrt{(2\pi/\omega)},$$

and to call $\gamma_{11}(\omega)$ and $\rho_1(\omega)$ the two simple gamma modular forms, the latter being sometimes called the simple Stirling modular form. We shall see that there exist three double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2),\ \gamma_{22}(\omega_1, \omega_2) \text{ and } \rho_2(\omega_1, \omega_2)$$

of exactly analogous nature.

§ 26. We proceed now to connect the function $\Gamma_2(z\,|\,\omega_1, \omega_2)$ with Alexeiewsky's function $G(z\,|\,\tau)$, some of whose properties were investigated in "The Genesis of the Double Gamma Functions."

In the first place, we take $\tau = \omega_2/\omega_1$, and then we have

$$G\left(\frac{z}{\omega_1}\middle|\tau\right) = e^{a\frac{z}{\omega_2} + b\frac{z}{2\omega_2^2}} \cdot \frac{z}{\omega_2} \cdot \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$,

and wherein

$$a = \frac{\tau}{z} \log 2\pi\tau + \tfrac{1}{2} \log \tau - \gamma\tau - C(\tau),$$

$$b = -\tau \log \tau - \frac{\pi^2\tau^2}{6} - \tau^2 D(\tau).$$

We also have

$$\Gamma_2^{-1}(z) = e^{\gamma_{21}\frac{z^2}{2} + z\gamma_{22}} \cdot z \cdot \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

so that on comparison of the two products we find

$$\Gamma_2^{-1}(z) = \mathrm{K} e^{\alpha z^2 + \beta z} \mathrm{G}\left(\frac{z}{\omega_1}\middle|\tau\right),$$

where α, β, and K are suitable functions of ω_1 and ω_2.

Now $\mathrm{G}(z|\tau)$ satisfies the difference equation

$$\mathrm{G}(z|\tau) = \Gamma^{-1}\left(\frac{z}{\tau}\right)\mathrm{G}(z+1|\tau),$$

and hence $\mathrm{G}\left(\frac{z}{\omega}\middle|\tau\right)$ satisfies the relation

$$f(z+\omega_1) = \omega_2^{1-\frac{z}{\omega_2}}\Gamma_1(z|\omega_2)f(z).$$

Hence a solution of the difference equation

$$f(z+\omega_1) = \surd(\omega_2/2\pi) \cdot \Gamma_1(z|\omega_2)f(z)\,e^{-2m\pi\iota\mathrm{S}_1'(z|\omega_2)}$$

is $$\mathrm{T}(z) = \omega_2^{\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_2} - \frac{z}{2\omega_1}}(2\pi)^{-\frac{z}{2\omega_1}}\mathrm{G}\left(\frac{z}{\omega_1}\middle|\tau\right)e^{-2m\pi\iota[{}_2\mathrm{S}_1'(z|\omega_1,\omega_2) - {}_2\mathrm{S}_1'(0)]} \text{ (by § 3).}$$

And it is evident that the coefficient of $2m\pi\iota$ in the last exponential may be written ${}_2\mathrm{S}_0(z|\omega_1,\omega_2)$.

The general solution of the difference equation is

$$\Gamma_2^{-1}(z) \times p(z|\omega_1),$$

where $p(z|\omega_1)$ is a function of z simply periodic of period ω_1.

Hence $$\frac{\Gamma_2^{-1}(z)}{\mathrm{T}(z)} = p(z|\omega_1).$$

But $\frac{\Gamma_2^{-1}(z)}{\mathrm{T}(z)}$ has been seen to be an expression of the form $\mathrm{K}e^{\alpha z^2 + \beta z^2}$, where K, α, and β are independent of z. We thus have

$$e^{\alpha(z+\omega_1)^2 + \beta(z+\omega_1)} = e^{\alpha z_2^2 + \beta z},$$

so that $\alpha = 0$ and $\beta = \frac{2r\pi\iota}{\omega_1}$, where r is some integer.

Hence $$\Gamma_2^{-1}(z) = \mathrm{K}e^{\frac{2r\pi\iota z}{\omega_1}}(\omega_2 e^{-m\pi\iota})^{{}_2\mathrm{S}_0(z)}(2\pi)^{\frac{-z}{2\omega_1}}\mathrm{G}\left(\frac{z}{\omega_1}\middle|\tau\right) \quad \ldots\ldots\ (1),$$

and since $$\underset{z=0}{\mathrm{L}t}[z\Gamma_2(z)] = 1, \quad \underset{z=0}{\mathrm{L}t}\left[\frac{z}{\mathrm{G}\left(\frac{z}{\omega_1}\middle|\tau\right)}\right] = \omega_2,$$

we have at once $$\mathrm{K} = \omega_2.$$

Now $G\left(\frac{z}{\omega_1}\middle|\tau\right)$ satisfies the equation*

$$\frac{f(z+\omega_2)}{\Gamma\left(\frac{z}{\omega_1}\right)f(z)} = \tau^{-\frac{z}{\omega_1}+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}},$$

where $\tau^{-\frac{z}{\omega_1}+\frac{1}{2}} = e^{\left(-\frac{z}{\omega_1}+\frac{1}{2}\right)\log\tau}$, and the principal value of $\log\tau$ is to be taken. (The same remark, of course, applies to every many-valued expression of this nature which occurs in the course of the investigation.)

Employing the relation (1) in conjunction with this equation and the equation

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\mid\omega_1)}{\sqrt{(2\pi/\omega_2)}}\, e^{-2m'\pi\iota\, S_1(z\mid\omega_1)},$$

we obtain

$$\Gamma_1(z\mid\omega_1)\,\omega_1^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\; e^{-2m'\pi\iota\, S_1'(z\mid\omega_1)}$$

$$= e^{2r\pi\iota\tau}\,\omega_2^{\frac{z}{\omega_2}-\frac{1}{2}}(2\pi)^{-\frac{\omega_2}{2\omega_1}}\quad e^{-2m\pi\iota\left(\frac{z}{\omega_1}-\frac{1}{2}\right)}\Gamma(z\mid\omega_1)\omega_1^{1-\frac{z}{\omega_1}}\,\tau^{-\frac{z}{\omega_1}+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{\tau}},$$

which reduces to

$$e^{2(m-m')\pi\iota\, S_1'(z\mid\omega_1)} = e^{2r\pi\iota\tau + S'_1(z\mid\omega_1)[\log\omega_2 - \log\omega_1 - \log\tau]},$$

But we have seen (§ 22) that

$$\log\omega_2 - \log\omega_1 - \log\tau = 2(m-m')\pi\iota$$

for $m - m' = 0$, unless the difference of the amplitudes of ω_1 and ω_2 is greater than π, in which case $m - m' = \pm 1$, according as $-I\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative.

We thus find $r = 0$, and incidentally we obtain a valuable verification of our results.

And now finally

$$\Gamma_2^{-1}(z) = \omega_2(2\pi)^{-\frac{z}{2\omega_1}}\left(\omega_2 e^{-2m\pi\iota}\right)^{{}_2S_0(z)}\, G\left(\frac{z}{\omega_1}\middle|\tau\right),$$

the relation between the two forms of double gamma functions.

§ 27. From the relation just found we may at once express the gamma modular constants $C(\tau)$ and $D(\tau)$ of the former theory in terms of $\gamma_{22}(\omega_1, \omega_2)$ and $\gamma_{21}(\omega_1, \omega_2)$ respectively.

For we have

$$G\left(\frac{z}{\omega_1}\middle|\tau\right) = e^{a\frac{z}{\omega_2}+b\frac{z^2}{2\omega_2^2}}\cdot\frac{z}{\omega_2}\cdot\prod_{m_1=0}^{\infty}\prod_{m_2=0}^{\infty}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right],$$

where

$$a = \frac{\tau}{2}\log 2\pi\tau + \tfrac{1}{2}\log\tau + \gamma\tau - C(\tau),$$

$$b = -\tau\log\tau - \frac{2\pi\tau^2}{6} - \tau^2 D(\tau),$$

and also

$$\Gamma_2^{-1}(z) = e^{2\,\gamma_{21}+z\gamma_{22}}\cdot z\cdot\prod_{m_1=0}^{\infty}\prod_{m_2=0}^{\infty}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right]$$

* "Genesis of the Double Gamma Functions," § 10.

Substituting in the relation

$$\Gamma_2^{-1}(z) = (2\pi)^{-\frac{z}{2\omega_1}\omega_2} \cdot (\omega_2 e^{-2m\pi\iota})^{{}_2S_0(z)} \cdot G\left(\frac{z}{\omega_1}\middle|\tau\right),$$

we find

$$\frac{z^2}{2}\gamma_{21}(\omega_1, \omega_2) + z\gamma_{22}(\omega_1, \omega_2) = -\frac{z}{2\omega_1}\log 2\pi + \left(\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_1} - \frac{z}{2\omega_2} + 1\right)\log \omega_2$$

$$- 2m\pi\iota\left(\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_1} - \frac{z}{2\omega_2}\right) + a\frac{z}{\omega_2} + b\frac{z^2}{2\omega_2^2}.$$

And hence equating coefficients of z and z^2 we find

$$\frac{\gamma_{21}(\omega_1, \omega_2)}{2} = \frac{1}{2\omega_1\omega_2}\{\log \omega_2 - 2m\pi\iota\} + \frac{b}{2\omega_2^2},$$

$$\gamma_{22}(\omega_1, \omega_2) = -\frac{1}{2\omega_1}\{\log 2\pi\omega_2 - 2m\pi\iota\} - \frac{1}{2\omega_2}\{\log \omega_2 + 2m\pi\iota\} + \frac{a}{\omega_2}.$$

Thus*

$$D(\tau) = -\omega_1^2\gamma_{21}(\omega_1, \omega_2) + \frac{\omega_1}{\omega_2}\{\log \omega_2 - \log \tau\} - \frac{\pi_2}{6} - \frac{\omega_1}{\omega_2}2m\pi\iota,$$

$$C(\tau) = -\omega_1\gamma_{22}(\omega_1, \omega_2) - \left(\tfrac{1}{2} + \frac{\omega_1}{2\omega_2}\right)\{\log \omega_2 - \log \tau - 2m\pi\iota\} + \gamma,$$

which are the relations required.

Since $$\log \omega_2 - \log \tau - 2m\pi\iota = \log \omega_1 - 2m'\pi\iota,$$

we may evidently write these relations in the form

$$D(\tau) = -\omega^2{}_1\gamma_{21}(\omega_1, \omega_2) + \frac{\omega_1}{\omega_2}\{\log \omega_1 - 2m'\pi\iota\} - \frac{\pi_2}{6},$$

$$C(\tau) = -\omega_1\gamma_{22}(\omega_1, \omega_2) - \left(\tfrac{1}{2} + \frac{\omega_1}{2\omega_2}\right)\{\log \omega_1 - 2m'\pi\iota\} + \gamma.$$

§ 28. We will now show that, when the parameters ω_1 and ω_2 are equal, we have

$$\gamma_{21}(\omega, \omega) = \frac{1}{\omega^2}\left[\log \omega - 1 - \gamma - \frac{\pi^2}{6}\right],$$

and $$\gamma_{22}(\omega, \omega) = \frac{1}{\omega}[\gamma - \tfrac{1}{2} - \log \omega].$$

By the definition formula of $\gamma_{21}(\omega_1, \omega_2)$, we at once have

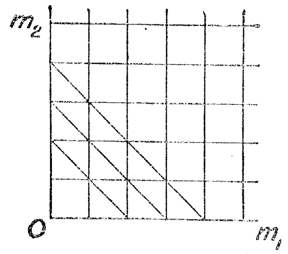
$$\omega^2\gamma_{21}(\omega, \omega) = -\underset{n=\infty}{\mathrm{Lt}}\left[\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{1}{(m_1+m_2)^2} - \log\frac{n\omega}{2}\right].$$

Group together all terms for which $m_1 + m_2 = \epsilon$, and we have

$$\sum^{n}\sum^{n}{}' \frac{1}{(m_1+m_2)^2} = \sum_{\epsilon=1}^{n}\frac{\epsilon+1}{\epsilon^2} + \frac{n}{(n+1)^2} + \frac{n-1}{(n+2)^2} + \ldots + \frac{1}{(n+n)^2},$$

* "Genesis of the Double Gamma Functions," § 6.

for we may suppose that the terms are represented by the corners of the small squares into which the positive quadrant is divided—in the new grouping we take together all terms lying on a line equally inclined to the two axes.



Thus
$$\sum_0^n \sum_0^{n}{}' \frac{1}{(m_1 + m_2)} = \sum_{\epsilon=1}^{n} \left(\frac{1}{\epsilon} + \frac{1}{\epsilon^2}\right) + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+n)^2}$$
$$+ \frac{n-1}{(n+1)^2} + \frac{n-2}{(n+2)^2} + \ldots + \frac{n-n}{(n+n)^2}.$$

So that when n is very large
$$\sum_0^n \sum_0^{n}{}' \frac{1}{(m_1 + m_2)^2} = \log n + \gamma + \frac{\pi^2}{6} + \int_0^1 \frac{1-x}{(1+x)^2} dx$$
$$+ \text{ terms which vanish when } n \text{ becomes infinite}$$
$$= \log n + \gamma + \frac{\pi_2}{6} + 1 - \log 2 + \text{similar terms.}$$

Hence
$$\gamma_{21}(\omega, \omega) = \frac{1}{\omega^2}\left[\log \omega - \frac{\pi_2}{6} - 1 - \gamma\right],$$
which is the first relation,

In the second place we have, when ω_1 and ω_2 are equal,
$$\omega\gamma_{22}(\omega, \omega) = \mathop{\mathrm{L}t}_{n=\infty}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{m_1 + m_2} - 2n \log 2 - \log 2n\omega\right],$$
and by the same method of reasoning as before
$$\sum_0^n \sum_0^{n}{}' \frac{1}{m_1 + m_2} = \sum_{\epsilon=1}^{n} \frac{\epsilon + 1}{\epsilon} + \frac{n}{n+1} + \frac{n-1}{n+2} + \ldots + \frac{1}{n+n},$$
$$= \sum_{\epsilon=1}^{n} \frac{1}{\epsilon} + n + \frac{1}{n+1} + \ldots + \frac{1}{n+n}$$
$$+ n\left\{2\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right) - \frac{n}{n}\right\}$$
$$= n + \log n + \gamma + \log 2 + 2n \log 2 - \tfrac{1}{2} - n,$$
neglecting terms which vanish when n is infinite, and thus
$$\gamma_{22}(\omega, \omega) = \frac{1}{\omega}\{\gamma - \tfrac{1}{2} - \log \omega\}.$$

It is an interesting piece of work to show that these results accord with those previously obtained for the G function of parameter unity.

§ 29. We proceed now to write down the expansion of $\log \Gamma_2(z)$ and the first few of the derived functions $\psi_2^{(r)}(z)$ in the vicinity of $z = 0$.

We have, by definition,

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{(z+\Omega)^3}, \text{ where } \Omega = m_1\omega_1 + m_2\omega_2,$$

Since the series on the right-hand side is absolutely convergent, we may expand in the form

$$\psi_2^{(3)}(z) = -\frac{2}{z^3} - 2\left\{\sum_0^\infty \sum{}' \frac{1}{\Omega^3} - \sum_0^\infty \sum{}' \frac{3z}{\Omega^4} + \sum_0^\infty \sum{}' \frac{3 \, . \, 4}{1 \, . \, 2} \frac{z^2}{\Omega^5} \quad \ldots \ldots \right\}.$$

Hence, integrating,

$$\psi_2^{(2)}(z) = \frac{1}{z^2} - \gamma_{21}(\omega_1, \omega_2) - 2\sum_0^\infty \sum{}' \frac{z}{\Omega^3} + 3\sum_0^\infty \sum{}' \frac{z^2}{\Omega^4} - 4\sum_0^\infty \sum{}' \frac{z^3}{\Omega^5} + \ldots \ldots$$

the constant being determined by making $z = o$.

Thus integrating again, and determining the constant in the same manner,

$$\psi_2^{(1)}(z) = -\frac{1}{z} - \gamma_{21}(\omega_1, \omega_2) z - \gamma_{22}(\omega_1, \omega_2) - \sum_0^\infty \sum{}' \frac{z}{\Omega^3} + \sum_0^\infty \sum{}' \frac{z^3}{\Omega^4} - \ldots \ldots$$

and finally,

$$\log \Gamma_2(z) = -\log z - z\gamma_{22} - \frac{z^2\gamma_{21}}{2} - \sum_0^\infty \sum{}' \frac{z^3}{3\Omega^3} + \sum_0^\infty \sum{}' \frac{z^4}{4\Omega^4} - \sum_0^\infty \sum{}' \frac{z^5}{5\Omega^5} \quad \ldots \ldots$$

the expansion holding good within a circle of radius just less than the least value of

$$|m_1\omega_1 + m_2\omega_2| \quad \left.\begin{matrix} m_1 = 0, 1, 2, \ldots \infty \\ m_2 = 0, 1, 2, \ldots \infty \end{matrix}\right\} \qquad \left.\begin{matrix} 0 \\ 0 \end{matrix}\right\} \text{excluded.}$$

We note that by EISENSTEIN's Theorem each coefficient in the series is an absolutely convergent series.

§ 30. We proceed now to the expressions for the double gamma functions as simply infinite products of simple gamma functions.

Consider the product

$$P(z) = \Gamma_1(z \mid \omega_1) \prod_{m_2=1}^{\infty} \left[\frac{\Gamma_1(z + m_2\omega_2 \mid \omega_1)}{\Gamma_1(m_2\omega_2 \mid \omega_1)} e^{-\psi_1^{(1)}(m_2\omega_2 \mid \omega_1) - \frac{z^2}{2}\psi_1^{(2)}(m_2\omega_2 \mid \omega_1)}\right].$$

The typical term may be written

$$e^{\frac{z^3}{3!}\psi_1^{(3)}(m_2\omega_2 \mid \omega_1) + \frac{z^4}{4!}\psi_1^{(4)}(m_2\omega_2 \mid \omega_1) + \ldots}$$

and the series $\sum_{m_2=1}^{\infty} \psi_1^{(r)}(m_2\omega_2 \mid \omega_1)$ are absolutely convergent when $r \geq 3$. The product is therefore in general absolutely convergent.

Again, it has no finite zeros, for its zeros would be those of $\Gamma_1(z + m_2\omega_2 | \omega_1)$ for $m_2 = 0, 1, \ldots \infty$. And its poles are given by

$$z + m_1\omega_1 + m_2\omega_2 = 0 \quad \begin{cases} m_1 = 0, 1, \ldots \infty. \\ m_2 = 0, 1, \ldots \infty. \end{cases}$$

Thus $\dfrac{\Gamma_2(z)}{\mathrm{P}(z)}$ has no zeros or poles in the finite part of the plane.

Change now z into $z + \omega_1$, and we have

$$\frac{\mathrm{P}(z+\omega_1)}{\mathrm{P}(z)} = z \prod_{m_2=1}^{\infty} \left[\left(1 + \frac{z}{m_2\omega_2}\right) e^{-\frac{z}{m_2\omega_2}} \right]$$
$$\times \prod_{m_2=1}^{\infty} \left[m_2\omega_2 \, . \, e^{-\omega_1 \psi_1^{(1)}(m_2\omega_2 | \omega_1) - \frac{2\omega_1 z + \omega_1^2}{2} \psi_1^{(2)}(m_2\omega_2 | \omega_1) + \frac{z}{m_2\omega_2}} \right].$$

Now the product last written must be convergent, for all other terms of the identity are finite for finite values of $|z|$, and this product is evidently of the form e^{pz+q}, where p and q are functions of ω_1 and ω_2 only.

Hence we must have

$$\mathrm{P}(z + \omega_1) = \mathrm{P}(z)\, \Gamma_1^{-1}(z | \omega_2)\, e^{\alpha z + \beta}.$$

Now we have proved that

$$\Gamma_2(z + \omega_1) = \Gamma_2(z)\, \Gamma_1^{-1}(z | \omega_2) \, . \left(\frac{2\pi}{\omega_2}\right)^{\frac{1}{2}} e^{2m\pi\iota\left(\frac{z}{\omega_2} - \frac{1}{2}\right)}.$$

Thus if we put

$$f(z) = \frac{\Gamma_2(z)}{\mathrm{P}(z)},$$

we shall have

$$\frac{f(z+\omega_1)}{f(z)} = e^{\alpha_1 z + \beta_1},$$

and similarly we shall have

$$\frac{f(z+\omega_2)}{f(z)} = e^{\alpha_2 z + \beta_2}.$$

Hence $f(z)$ is a doubly periodic function of the third kind, with no finite zeros or poles.

Thus we must have

$$f(z) = \mathrm{C}e^{\mathrm{A}z^2 \times \mathrm{B}z},$$

for in HERMITE's expression of such a function the σ functions are each associated with a finite non-congruent zero.*

To determine C we put $z = 0$, and obtain

$$\mathrm{C} = \underset{z=0}{\mathrm{L}t} \left[\frac{\Gamma_2(z)}{\Gamma_1(z | \omega_1)} \right] = 1.$$

Differentiating logarithmically, the identity

$$\Gamma_2(z_1 | \omega, \omega_2) = \mathrm{P}(z)\, e^{\mathrm{A}z^2 + \mathrm{B}z},$$

* FORSYTH, 'Theory of Functions,' § 142.

we find on making $z = 0$,

$$-\gamma_{22}(\omega_1, \omega_2) = B + \frac{1}{\omega_1}(\log \omega_1 - \gamma),$$

and therefore $$B = -\gamma_{22}(\omega_1, \omega_2) + \frac{\gamma}{\omega_1} - \frac{\log \omega_1}{\omega_1}.$$

Differentiating again, we have

$$\psi_2^{(2)}(z \mid \omega_1, \omega_2) = 2A + \psi_1^{(2)}(z \mid \omega_1) + \sum_{m_2=1}^{\infty} [\psi_1^{(2)}(z + m_2\omega_2 \mid \omega_1) - \psi_1^{(2)}(m_2\omega_2 \mid \omega_1)].$$

Again, making $z = 0$, we find

$$-\gamma_{21}(\omega_1, \omega_2) = 2A,$$

or $$A = -\tfrac{1}{2}\gamma_{21}(\omega_1, \omega_2).$$

Finally, then, we have

$$\Gamma_2(z \mid \omega_1, \omega_2) = e^{-\frac{\gamma_{21}z^2}{2} - z\left\{\gamma_{22} + \frac{1}{\omega_1}\log \omega_1 - \frac{\gamma}{\omega_1}\right\}} \times \Gamma(z \mid \omega_1)$$
$$\times \prod_{m_2=1}^{\infty} \left[\frac{\Gamma_1(z + m_2\omega_2 \mid \omega_1)}{\Gamma_1(m_2\omega_2 \mid \omega_1)} e^{-z\psi_1^{(1)}(m_2\omega_2 \mid \omega_1) - \frac{z^2}{2}\psi_1^{(2)}(m_2\omega_2 \mid \omega_1)}\right].$$

This formula is equivalent to the one obtained in the "Genesis of the Double Gamma Functions," § 2.

It is an interesting verification to actually transform the one formula into the other, making use of the relations established between $\gamma_{21}(\omega_1, \omega_2)$, $\gamma_{22}(\omega_1, \omega_2)$, $C(\tau)$ and $D(\tau)$.

On account of the symmetry of the present functions, the formula corresponding to that given in § 8 of the "Genesis of the Double Gamma Functions" may be written

$$\Gamma_2(z \mid \omega_1, \omega_2) = e^{-\gamma_{21}\frac{z}{2} - z\left\{\gamma_{22} + \frac{1}{\omega_2}\log \omega_2 - \frac{\gamma}{\omega_2}\right\}} \cdot \Gamma_1(z \mid \omega_2)$$
$$\times \prod_{m_1=1}^{\infty} \left[\frac{\Gamma_1(z + m_1\omega_1 \mid \omega_2)}{\Gamma_1(m_1\omega_1 \mid \omega_2)} e^{-z\psi_1^{(1)}(m_1\omega_1 \mid \omega_2) - \frac{z^2}{2}\psi_1^{(2)}(m_1\omega_1 \mid \omega_2)}\right].$$

The product formulæ just obtained correspond to the expression of the σ function as an infinite product of circular functions.

Such a circumstance, of course, at once prompts us to try and find a formula corresponding to the expression of the σ function as a *sum* of exponential functions. But it is readily seen that such is an impossibility. We cannot express the double gamma function as a sum of an infinite series of simple gamma functions of varying arguments.

It is this fact, combined with the absence of any quasi-addition theorem for the double gamma functions, which precludes the possibility of any collection of formulæ rivalling in number and elegance those of the doubly periodic functions.

§ 31. We proceed now to express WEIERSTRASS' elliptic functions in terms of double gamma functions.

In WEIERSTRASS' notation of elliptic functions we have

$$\wp'(z) = -2 \sum_{m_1}^{\infty} \sum_{m_2} \frac{1}{(z+\Omega)^3},$$
$$\scriptstyle -\infty$$

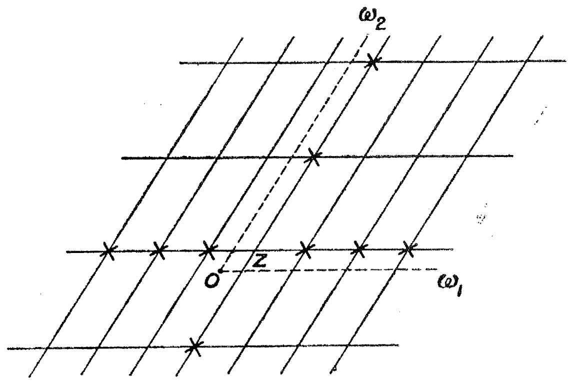
where $\Omega = m_1\omega_1 + m_2\omega_2$, and ω_1 and ω_2 are the periods of $\wp(z)$.

Now by definition

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z+\Omega)^3}.$$

Representing the various terms by the corners of the parallelograms of the figure, we readily see that

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) + \psi_2^{(3)}(z \mid -\omega_1, \omega_2) + \psi_2^{(3)}(z \mid \omega_1, -\omega_2) + \psi_2^{(3)}(z \mid -\omega_1, -\omega_2)$$
$$= -2\left[\sum_{-\infty}^{\infty}\sum \frac{1}{(z+\Omega)^3} + \sum_{-\infty}^{\infty} \frac{1}{(z+m_1\omega_1)^3} + \sum_{-\infty}^{\infty} \frac{1}{(z+m_2\omega_2)^3} + \frac{1}{z^3}\right].$$



Hence, using the natural summation $\Sigma\psi_2^{(3)}(z \mid \pm\omega_1, \pm\omega_2)$ to express the left-hand side of this relation, we have

$$\Sigma\psi_2^{(3)}(z \mid \pm\omega_1, \pm\omega_2) = \wp'(z) + \Sigma\psi_1^{(3)}(z \mid \pm\omega_1) + \Sigma\psi_1^{(3)}(z \mid \pm\omega_2) + \frac{2}{z^3},$$

and therefore, on integration,

$$\wp(z) = \Sigma\psi_2^{(2)}(z \mid \pm\omega_1, \pm\omega_2) - \Sigma\psi_1^{(2)}(z \mid \pm\omega_1) - \Sigma\psi_1^{(2)}(z \mid \pm\omega_2) + \frac{1}{z^2} + \nu,$$

where ν is constant with respect to z.

Evidently

$$\nu = \Sigma\gamma_{21}(\pm\omega_1, \pm\omega_2) + 2\sum_{1}^{\infty} \frac{1}{(n\omega_1)^2} + 2\sum_{1}^{\infty} \frac{1}{(n\omega_2)^2}.$$

Now

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\frac{1}{\omega_1\omega_2}[\log n\omega_2 + \log n\omega_1 - \log n(\omega_1+\omega_2)] - \sum_{m_1=0}^{n} \sum_{m_2=0}^{n'} \frac{1}{(m_1\omega_1 + m_2\omega_2)^2}\right],$$

where the principal values of the logarithms are to be taken.

And hence

$$\nu = \mathop{\mathrm{L}t}_{n=\infty}\left[-\sum_{m_1=-n}^{n}\sum_{m_2=-n}^{n}\frac{1}{\Omega^2}+\frac{1}{\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_1+\omega_2)+\log[-(\omega_1+\omega_2)]\\ -\log(\omega_1-\omega_2)-\log[-(\omega_1-\omega_2)]\end{array}\right\}\right].$$

Now, as may be readily seen by examining the different possible cases in a diagram,

$$\log(\omega_1+\omega_2)+\log[-(\omega_1+\omega_2)]-\log(\omega_1-\omega_2)-\log[-(\omega_1-\omega_2)] = 2\log\frac{\omega_1+\omega_2}{\omega_1 \backsim \omega_2},$$

where that value of $\omega_1 \backsim \omega_2$ is to be taken which is on the same side of the real axis as $\omega_1 + \omega_2$.

With this proviso,

$$\nu = -\sum_{-\infty}\sum{}'\frac{1}{\Omega^2}+\frac{2}{\omega_1\omega_2}\log\frac{\omega_1+\omega_2}{\omega_1\backsim\omega_2},$$

the infinities in the double summation being equal in absolute magnitude.

§ 32. We may now express WEIERSTRASS' ζ function in terms of derivatives of simple and double gamma functions. For on integrating the relation obtained in the previous paragraph we find,

remembering that $\qquad \frac{d}{dz}\zeta(z) = -\wp(z),$*

$$-\zeta(z) = \Sigma\psi_1^{(1)}(z\,|\pm\omega_1,\pm\omega_2) - \Sigma\psi_1^{(1)}(z\,|\pm\omega_1) - \Sigma\psi_1^{(1)}(z\,|\pm\omega_2) - \frac{1}{z} + \nu z + \mu,$$

where μ is constant with respect to z.

Making then $z = 0$, we find

$$o = -\Sigma\gamma_{22}(\pm\omega_1,\pm\omega_2) - \frac{1}{\omega_1}\{\log(\omega_1)-\gamma\} + \frac{1}{\omega_1}\log(-\omega_1)-\gamma\}$$
$$-\frac{1}{\omega_2}\{\log\omega_2-\gamma\}+\frac{1}{\omega_2}\{\log(-\omega_2)-\gamma\}+\mu.$$

Now by § 23,

$$\Sigma\gamma_{22}(\pm\omega_1,\pm\omega_2) = \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_1+\omega_2)-\log\omega_1-\log\omega_2\\ -\log[-(\omega_1+\omega_2)]+\log(-\omega_1)+\log(-\omega_2)\end{array}\right\}$$
$$+\frac{\omega_1-\omega_2}{2\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_2-\omega_1)-\log(-\omega_1)-\log\omega_2\\ -\log(\omega_1-\omega_2)+\log\omega_1+\log(-\omega_2)\end{array}\right\}$$
$$=\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_2}\right)\{\log(\omega_1+\omega_2)-\log[-(\omega_1+\omega_2)]\}$$
$$+\frac{1}{\omega_1}\{\log(-\omega_1)-\log\omega_1\}+\frac{1}{\omega_2}\{\log(-\omega_2)-\log\omega_2\}$$
$$-\left\{\frac{1}{2\omega_1}-\frac{1}{2\omega_2}\right\}\{\log(\omega_2-\omega_1)-\log(\omega_1-\omega_2)\}.$$

* JORDAN, 'Cours d'Analyse,' 2nd edition, p. 347. Note that JORDAN uses $2\omega_1$ and $2\omega_2$ instead of ω_1 and ω_2 of this paper.

Hence $$\mu = \left(\frac{1}{2\omega_1} + \frac{1}{2\omega_2}\right)\{\log(\omega_1 + \omega_2) - \log[-(\omega_1 + \omega_2)]\}$$
$$- \left\{\frac{1}{2\omega_1} - \frac{1}{2\omega_2}\right\}\{\log(\omega_2 - \omega_1) - \log(\omega_1 - \omega_2)\}.$$

Now $$\log(\omega_1 + \omega_2) - \log[-(\omega_1 + \omega_2)] = \pm \pi\iota,$$

according as $I(\omega_1 + \omega_2)$ is positive or negative, and

$$\log(\omega_2 - \omega_1) - \log(\omega_1 - \omega_2) = \mp \pi\iota,$$

according as $I(\omega_1 - \omega_2)$ is positive or negative. Therefore the values of μ are given in the following table :—

	$I(\omega_1 + \omega_2)$ positive.	$I(\omega_1 + \omega_2)$ negative.
$I(\omega_1 - \omega_2)$ positive	$\mu = \frac{\pi\iota}{\omega_1}$	$\mu = -\frac{\pi\iota}{\omega_2}$
$I(\omega_1 - \omega_2)$ negative	$\mu = \frac{\pi\iota}{\omega_2}$	$\mu = -\frac{\pi\iota}{\omega_1}$

In other words,

if $|I(\omega_1)| > |I(\omega_2)|$, $\mu = \pm \pi\iota/\omega_1$, the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative, and

if $|I(\omega_2)| > |I(\omega_1)|$, $\mu = \pm \pi\iota/\omega_2$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

§ 33. The expression for $\sigma(z)$ in terms of simple and double gamma functions is now immediate.

For on integrating the result of the previous paragraph

$$- \log \sigma(z) = \rho + \mu z + \frac{\nu z^2}{2} - \log z + \Sigma \log \Gamma_2(z \mid \pm \omega_1, \pm\omega_2) - \Sigma \log \Gamma_1(z \mid \pm\omega_1)$$
$$- \Sigma \log \Gamma_1(z \mid \pm\omega_2),$$

ρ being the constant of integration.

Make now $z = 0$, and we at once see that $\rho = 0$.

Hence $$\sigma(z) = e^{-\mu z - \nu\frac{z^2}{2}} \cdot z \cdot \frac{\Pi\Gamma_2^{-1}(z \mid \pm \omega_1, \pm\omega_2)}{\Pi\Gamma_1^{-1}(z \mid \pm \omega_1)\Pi\Gamma_1^{-1}(z \mid \pm \omega_2)},$$

and in this expression

$$\nu = - \sum_{-\infty}^{\infty}\sum{}' \frac{1}{\Omega^2} + \frac{z}{\omega_1\omega_2} \log \frac{\omega_1 + \omega_2}{\omega_1 \backsim \omega_2},$$

the infinities in the double summation being equal in absolute magnitude, and that value of $\omega_1 \backsim \omega_2$ being taken which is on the same side of the real axis as $\omega_1 + \omega_2$; while

$\mu = \pm \frac{\pi\iota}{\omega_1}$ if $|\, I(\omega_1)| > |\, I(\omega_2)|$, the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative,

and $\mu = \pm \frac{\pi\iota}{\omega_2}$ if $|\, I(\omega_2)\,| > |I(\omega_1)|$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

§ 34. By means of the preceding paragraphs we may now at once prove WEIERSTRASS' relation*

$$\omega_2\eta_1 - \omega_1\eta_2 = \pm\, 2\pi\iota,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative, where η_1 and η_2 are determined by the relations

$$\zeta(z + \omega_1) = \zeta(z) + \eta_1$$
$$\zeta(z + \omega_2) = \zeta(z) + \eta_2.$$

Take the expression for $\zeta(z)$ given in § 32, and we find by use of the formulæ of § 22 that

$$\zeta(z) - \zeta(z + \omega_1) = \frac{2\pi\iota}{\omega_2}(m - m_1 - m_2 + m_3) + \nu\omega_1,$$

where

$m_1 = 0$, unless $\left.\begin{matrix}\omega_1 \text{ does} \\ \omega_1 - \omega_2 \text{ does not}\end{matrix}\right\}$ lie within the region bounded by axes from the origin to -1 and ω_2, in which case $m_1 = \pm 1$, according as $-I(\omega_2)$ is positive or negative, and m_2 and m_3 are obtained by changing the signs of (i) both ω_1 and ω_2 and (ii) ω_1 respectively in this formula.

Thus $$\frac{\eta_1}{\omega_1} = \sum_{-\infty}^{\infty}\sum{}' \frac{1}{\Omega^2} - \frac{2}{\omega_1\omega_2}\log\frac{\omega_1 + \omega_2}{\omega_1 \backsim \omega_2} - \frac{2\pi\iota}{\omega_1\omega_2}[m - m_1 - m_2 + m_3],$$

the infinities in the double summation being equal in absolute magnitude, and that value of $\omega_1 \backsim \omega_2$ being taken which is on the same side of the real axis as $(\omega_1 + \omega_2)$.

And, similarly,

$$\zeta(z + \omega_2) - \zeta(z) = \eta_2,$$

where $$\frac{\eta_2}{\omega_2} = \sum_{-\infty}^{\infty}\sum{}' \frac{1}{\Omega^2} - \frac{2}{\omega_1\omega_2}\log\frac{\omega_1 + \omega_2}{\omega_1 \backsim \omega_2} + \frac{2\pi\iota}{\omega_1\omega_2}[-m' + m_1' + m_2' - m_3'],$$

where m' has its usual meaning, and m_r' is obtained from m' just as is m_r from m.

Hence $$\frac{\eta_1}{\omega_1} - \frac{\eta_2}{\omega_2} = \frac{2\pi\iota}{\omega_1\omega_2}[-(m - m') + (m_1 - m_1') + (m_2 - m_2') - (m_3 - m_3')].$$

* JORDAN, 'Cours d'Analyse,' p. 351; FORSYTH, 'Theory of Functions,' § 129. Again notice that each of the quantities η and ω is double that usually taken.

But we have seen in § 22 that, if the difference of amplitudes of ω_1 and ω_2 be greater than π,

$$m - m' = \pm 1,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_1}{\omega_2}\right)$ is positive or negative.

Similarly it may be proved that if the difference of amplitude of ω_1 and $-\omega_2$ is $> \pi$,

$$m_1 - m_1' = \pm 1, \text{ according as } I\left(\frac{\omega_2}{\omega_1}\right) \text{ is } +ve \text{ or } -ve.$$

And . . . . $-\omega_1$ and ω_2 is $> \pi$.

$$m_2 - m_2' = \pm 1 \quad . \quad . \quad . \quad . \quad I\left(\frac{\omega_2}{\omega_1}\right) \text{ is } +ve \text{ or } -ve;$$

while, finally, ... $-\omega_1$ and $-\omega_2$ is $> \pi$.

$$m_3 - m_3' = \mp 1 \quad . \quad . \quad . \quad . \quad I\left(\frac{\omega_3}{\omega_1}\right) \text{ is } +ve \text{ or } -ve.$$

In all other cases the differences between corresponding m's vanish.

But the difference of amplitude of one, and of only one, pair of the set

$$\omega_1, \omega_2; \ \omega_1, -\omega_2; \ -\omega_1, \omega_2; \ -\omega_1, -\omega_2$$

can be greater than π.

Hence $$-(m - m') + (m_1 - m_1') + (m_2 - m_2') - (m_3 - m_3')$$

must always equal ± 1.

And it is easy to see, by taking the particular cases which can arise, that the upper or lower sign must be taken according as $I(\omega_2/\omega_1)$ is positive or negative.

We have then finally

$$\frac{\eta_1}{\omega_1} - \frac{\eta_2}{\omega_2} = \pm \frac{2\pi\iota}{\omega_1\omega_2},$$

and therefore

$$\eta_1\omega_2 - \eta_2\omega_1 = \pm 2\pi\iota,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative.

§ 35. For brevity, we merely indicate the relation of the formulæ which we have found to the known relations:

$$\sigma(z + \omega_1) = -e^{\eta_1\left(z + \frac{\omega_1}{2}\right)}\sigma(z)$$

$$\sigma(z + \omega_2) = -e^{\eta_2\left(z + \frac{\omega_2}{2}\right)}\sigma(z).$$

By § 33

$$\frac{\sigma(z + \omega_1)}{\sigma(z)} = e^{-\mu\omega_1 - \nu\frac{2z\omega_1 + \omega_1^2}{2}} \cdot \frac{z + \omega_1}{z} \cdot \frac{\Pi\left\{\frac{\Gamma_2^{-1}(z + \omega_1 | \pm\omega_1, \pm\omega_2)}{\Gamma_2^{-}(z | \pm\omega_1, \pm\omega_2)}\right\}}{\Pi\frac{\Gamma_1^{-1}(z + \omega_1 | \pm\omega_1)}{\Gamma_1^{-1}(z | \pm\omega_1)}\ \Pi\frac{\Gamma_1^{-1}(z + \omega_1 | \pm\omega_2)}{\Gamma_1^{-1}(z | \pm\omega_2)}};$$

and by the previous paragraph, the expression on the right-hand side of this relation will evidently reduce to the form $e^{\eta_1 z + \delta_1}$, where δ_1 is some constant whose value may be readily seen to be given by

$$\frac{\delta_1 - \frac{1}{2}\eta_1\omega_1}{\omega_1} = -\mu + \pi\iota(m - m_3)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + \pi\iota(m_2 - m_1)\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right).$$

This value simplifies on detailed consideration.

When $|I(\omega_1)| > |I(\omega_2)|$,

$$\delta_1 - \tfrac{1}{2}\eta_1\omega_1 = \mp \pi\iota,$$

according as $I(\omega_1)$ is positive or negative.

When $|I(\omega_2)| > |I(\omega_1)|$, there are four subsidiary possibilities ;—

(α) When ω_1 lies within the axes to -1 and $-\omega_2$,
(β) ,, ω_1 ,, ,, ,, -1 and ω_2,
(γ) ,, $-\omega_1$,, ,, ,, -1 and $-\omega_2$,
and (δ) ,, $-\omega_1$,, ,, ,, -1 and ω_2.

In cases (α) and (δ) $\delta_1 - \frac{1}{2}\eta_1\omega_1 = \pm \pi\iota/\omega_1$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative ; and in cases (β) and (γ) the upper and lower signs are interchanged.

We thus see that in all cases

$$e^{\delta_1} = -e^{-\frac{1}{2}\eta_1\omega_2},$$

so that we have the required equation

$$\sigma(z + \omega_1) = -e^{\eta_1(z + \frac{1}{2}\omega_1)}\sigma(z).$$

Similarly we find $\quad \sigma(z + \omega_2) = -e^{\eta_2(z + \frac{1}{2}\omega_2)}\sigma(z).$

The verification of these results affords substantial proof of the general correctness of the signs which are involved in the work.

§ 36. It is interesting finally to notice that just as the gamma functions do not exist when $\tau = \omega_2/\omega_1$ is real and negative, so the elliptic functions demand that τ shall not be real.

The condition that τ must not be real and negative arose explicitly at several stages, and might have been predicted *à priori*.

For, when ω_2/ω_1 is real and negative, it is obvious that

$$\Omega = m_1\omega_1 + m_2\omega_2 \begin{cases} m_1 = 0, 1, \ldots, \infty & 0 \\ m_2 = 0, 1, \ldots, \infty & 0 \end{cases} \text{excluded.}$$

will have a zero value at least once.

And thus the function

$$\Gamma_2^{-1}(z) = e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}} \cdot z \cdot \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right)e^{-\frac{z}{\Omega} + \frac{z_2}{2\Omega^2}}\right]$$

will be infinite independently of z ; that is to say, it ceases to exist.

For all other values of ω_1 and ω_2 the function $\Gamma_2^{-1}(z\,|\,\omega_1, \omega_2)$ exists. But the product

$$\Pi\Gamma_2(z\,|\pm\omega_1, \pm\omega_2),$$

and consequently $\sigma(z)$, will not exist when either ω_2/ω_1 or $-\omega_2/\omega_1$ is real and negative; that is, when τ is real. The criterion for the existence of multiple gamma functions (n-ple where n is greater than 2) is more intricate, and, as we know, n-ply periodic functions ($n > 2$) do not exist.*

PART III.

Contour Integrals connected with the Double Gamma Function. The Double Riemann Zeta Function.

§ 37. In the theory of the simple gamma function it was shown that the intervention of a definite integral, coupled with the theory of asymptotic approximations, it was possible to obtain contour and line integrals to express EULER's constant γ, and the logarithm of the simple gamma function and its derivatives. We now proceed to show that it is possible to extend the method thus previously employed so as to obtain expressions as contour and line integrals for the gamma modular constants γ_{21} and γ_{22}, and the logarithm of the double function and its derivatives. It will be noticed that when the real parts of ω_1 and ω_2 are positive, the numbers m and m' which intervened in Part II. vanish, and there is consequently a noteworthy simplification of the formulæ obtained. This simplification extends also to the definite integral expressions, and consequently we shall first investigate the theory in this simple case, proceeding subsequently to contour integrals of greater complexity. Finally we make use of an extension of MELLIN's method of defining the simple ζ function by a series instead of a contour integral, and we show that there is complete agreement between the formulæ obtained in the different ways.

§ 38. When the real parts of ω_1 and ω_2 are positive, and when in addition the real part of a is positive, we define the double Riemann ζ function

$$\zeta_2(s, a\,|\,\omega_1, \omega_2)$$

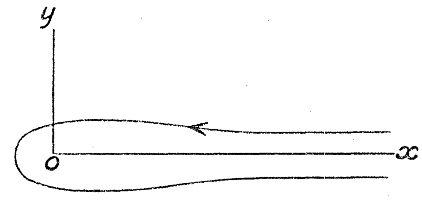
for all values, real or complex of s, by the integral

$$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-az}(-z)^{s-1}dz)}{(1-e^{\omega_1 z})(1-e^{-\omega_2 z})},$$

wherein $(-z)^{s-1} = e^{(s-1)\log(-z)}$, $\log(-z)$ being real when z is negative and being rendered uniform by a cut along the positive direction of the real axis. The integral is to be taken along a contour enclosing the origin (but no other pole of the

* The existence-criteria for functions which are substantially multiple gamma functions have been discussed by CRANI, 'Batt. Gior.,' vol. 29.

subject of integration), and the positive half of the real axis; and extending from $+\infty$ to $+\infty$ as in the figure.



Under the limitations specified the integral is, in general, finite. Moreover, by a theorem previously obtained,* we have under such limitations

$$\frac{\iota\Gamma(1-s)}{2\pi}\int(-z)^{s-1}e^{-(a+m_1\omega_1+m_2\omega_2)z}dz = \frac{1}{(a+m_1\omega_1+\omega_2\omega_2)^s},$$

the latter expression having its principal value.

And therefore

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+\omega_2\omega_2)^s}$$

$$=\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{1-e^{-(pn+1)\omega_1 z}}{1-e^{-\omega_1 z}}\cdot\frac{1-e^{-(qn+1)\omega_2 z}}{1-e^{-\omega_2 z}}e^{-az}(-z)^{s-1}dz$$

$$=\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[p(n+1)\omega_1+a]z}+e^{-[q(n+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$+\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[a+p(n+1)\omega_1+q(n+1)\omega_2]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz,$$

all the integrals being taken along the fundamental contour.

When n is a large positive integer, we proceed to throw the two integrals last written into the form of asymptotic series.

For this purpose consider the expansion obtained in § 16, which may be written

$$\frac{ze^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}=\frac{1}{\omega_1\omega_2 z}-{}_2S'_0(a+\omega_1)+\frac{{}_2S'_1(a+\omega_1)}{1!}z+\ldots$$

$$+\frac{(-)^{n-1}{}_2S'_n(a+\omega_1)}{n!}z^n+\ldots$$

We showed that this expansion is valid provided $|z|$ was less than the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|$, $\left|\frac{2\pi\iota}{\omega_2}\right|$,

Outside this circle the series diverges. But within the region bounded by lines going to infinity from the poles of

$$\frac{ze^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}-\frac{1}{\omega_1\omega_2 z},$$

* "Theory of the Gamma Function," §§ 22, 33 and 34.

the series will, in the language of M. BOREL,* be summable, that is to say, within this region (which is conveniently bounded by straight lines from the points $\pm \frac{2\pi\iota}{\omega_1}$, $\pm \frac{2\pi\iota}{\omega_2}$ which pass to infinity through the remaining poles) it is possible from the values of terms of the series at any point to obtain, by the employment of intermediate functions, a magnitude, independent of these particular intermediate functions, which is the value of the function at the point.

If then on any term $\alpha_n z^n$ of such a series we perform the operation which is expressed by

$$\frac{\iota\Gamma(1-s)}{2\pi}\int \alpha_n z^n (-z)^{s-1} dz,$$

we shall expect to obtain a quantity which is the n-th term of a possibly, and even probably, divergent sequence, which in turn is, by suitable operations, summable to the value which results from the performance of the fundamental operation on the function of which the original series is the expression.

Such considerations being understood to underlie the operations, we have

$$\frac{\iota\Gamma(1-s)}{2\pi}\int \frac{e^{-[p(n+1)\omega_1+a]z} + e^{-[q(m+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} (-z)^{s-1} dz,$$

$$= \frac{\iota\Gamma(1-s)}{2\pi}\int \frac{(-z)^{s-3}}{\omega_1\omega_2}\left[e^{-pn\omega_1 z} + e^{-qn\omega_2 z}\right] dz$$

$$+ \frac{\iota\Gamma(1-s)}{2\pi}\int (-z)^{s-2} e^{-pn\omega_1 z}\left\{\frac{\omega_1+\omega_2}{2\omega_1\omega_2} + \frac{a-\omega_2}{\omega_1\omega_2}\right\} dz$$

$$+ \frac{\iota\Gamma(1-s)}{2\pi}\int (-z)^{s-2} e^{-qn\omega_2 z}\left\{\frac{\omega_1+\omega_2}{2\omega_1\omega_2} + \frac{a-\omega_2}{\omega_1\omega_2}\right\} dz$$

$$+ \frac{\iota\Gamma(1+s)}{2\pi}\sum_{m=1}^{\infty}\int \frac{(-z)^{m+s-2}}{m\,!}\left\{{}_2S'_m(a+\omega_1)e^{-pn\omega_1 z} + {}_2S'_m(a+\omega_2)e^{-qn\omega_2 z}\right\} dz$$

$$= \frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-2}} + \frac{1}{(qn\omega_2)^{s-2}}\right\} - \frac{1}{s-1}\,\frac{\omega_1+\omega_2+2a}{2\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-1}} + \frac{1}{(qn\omega_2)^{s-1}}\right\}$$

$$+ \frac{1}{s-1}\left\{\frac{1}{\omega_1}\cdot\frac{1}{(pn\omega_1)^{s-1}} + \frac{1}{\omega_2}\frac{1}{(qn\omega_2)^{s-1}}\right\}$$

$$+ \sum_{m=1}^{\infty}(-)^{m-1}\frac{s\,.\,(s+1)\,.\,.\,.\,(s+m-2)}{1\,.\,2\,.\,.\,.\,m}\left\{\frac{{}_2S'_m(a+\omega_1)}{(pn\omega_1)^{m+s-1}} + \frac{{}_2S'_m(a+\omega_2)}{(qn\omega_2)^{m+s-1}}\right\}.$$

Now it may be readily deduced from the results obtained in Part I., that

$${}_2S_m{}'(a) = m\left[{}_2S_{m-1}(a+\omega_1) + {}_2B_m\right]$$

* BOREL, 'Liouville,' 5 Sér., vol. 2, pp. 103 *et seq.* 'Annales de École Normale Supérieure,' 3 Sér., vol. 6, pp. 1 *et seq.*

Hence we have

$$-\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[(pn+1)\omega_1+a]z}+e^{-[(qn+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$=-\frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-2}}+\frac{1}{(qn\omega_2)^{s-2}}\right\}+\frac{1}{s-1}\cdot\frac{2a+\omega_1+\omega_2}{\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-1}}\right.$$

$$\left.+\frac{1}{(qn\omega_2)^{s-1}}\right\}-\frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}}+\frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}-\left\{\frac{{}_2S_1'(a+\omega_1)}{(pn\omega_1)^s}-\frac{{}_2S_1'(a+\omega_2)}{(qn\omega_2)^s}\right\}$$

$$+\sum_{m=1}(-)^{m-1}\binom{s+m-1}{m}\left\{\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(pn\omega_1)^{m+s}}+\frac{{}_2S_m(a+\omega_2)+{}_2B_{m+1}}{(qn\omega_2)^{m+s}}\right\}$$

In an exactly similar manner we find, since

$${}_2S_m'(a+\omega_1+\omega_2)=m\left[{}_2S_{m-1}(a+\omega_1+\omega_2)+{}_2B_m\right],$$

that

$$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[a+(pn+1)\omega_1+(qn+1)\omega_2]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$=\frac{1}{(s-1)(s-2)\omega_1\omega_2}\cdot\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}}-\frac{2a+\omega_1+\omega_2}{(s-1)2\omega_1\omega_2}\cdot\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}$$

$$+\frac{{}_2S_1'(a+\omega_2+\omega_2)}{(pn\omega_2+qn\omega_2)^s}+\sum_{m=1}^{\infty}(-)^m\binom{s+m-1}{m}\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(pn\omega_1+qn\omega_2)^s}$$

If now we group together all the results which have been obtained, we find the asymptotic quality, true for all values of s, and for values of a, ω_1, and ω_2, whose real parts are positive,

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}=\zeta_2(s,a\,|\,\omega_1,\omega_2)$$

$$+\frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}}-\frac{1}{(pn\omega_1)^{s-2}}\quad\frac{1}{(qn\omega_2)^{s-2}}\right\}$$

$$-\frac{2a+\omega_1+\omega_2}{2(s+1)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}-\frac{1}{(pn\omega_1)^{s-1}}-\frac{1}{(qn\omega_2)^{s-1}}\right\}$$

$$-\frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}}+\frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}$$

$$+\frac{{}_2S_1'(a+\omega_1+\omega_2)}{(pn\omega_1+qn\omega_2)^s}-\frac{{}_2S_1'(a+\omega_1)}{(pn\omega_1)^s}-\frac{{}_2S_1'(a+\omega_2)}{(qn\omega_2)^s}$$

$$+\sum_{m=1}^{\infty}\frac{(-)^m}{n^{m+s}}\binom{m+s-1}{m}\left\{\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(p\omega_1+q\omega_2)^{m+s}}-\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(p\omega_1)^{s+m}}\right.$$

$$\left.-\frac{{}_2S_m(a+\omega_2)+{}_2B_{m+1}}{(q\omega_2)^{m+s}}\right\}\quad .\quad .\ \text{(A)}$$

It may readily be seen, just as for the case of a single parameter, that the series proceeding by powers of $\frac{1}{n}$ is a series of powers of a real variable, whose line of convergency is of zero length.

This series is summable by an almost evident modification of the application of BOREL's ideas, which was employed in the "Theory of the Gamma Function," § 38.

It is thus the asymptotic equivalent of the sum

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s};$$

and it satisfies POINCARÉ's* criterion for asymptotic equality that the difference between this sum and the first m terms of the series has its absolute value less than a quantity of order $\frac{1}{n^{m+s-2}}$.

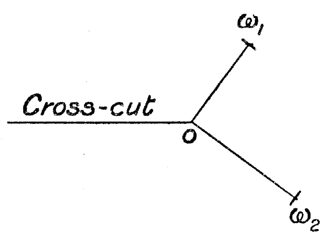
§ 39. The function $\zeta_2(s, a \mid \omega_1, \omega_2)$ has been defined, and the asymptotic equality (A) has been deduced only for the case in which the real parts of a, ω_1, and ω_2 are all positive.

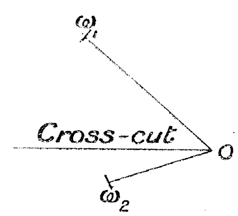
It is natural to try and use the equality to define the function for all values of a, ω_1, and ω_2.

In the first place, it is evident that when $\mathfrak{R}(\omega_1)$ and $\mathfrak{R}(\omega_2)$ are both positive, the equality (A) holds for all values of a, for the various terms of the sum and the equivalent asymptotic series are continuous for all but an enumerable number of values of a, ω_1, and ω_2. Hence in this case $\zeta_2(s, a \mid \omega_1, \omega_2)$ may be defined as the term independent of n in the equality.

So also when s is a real positive or negative integer, the equality will hold for all values of a, ω_1, and ω_2.

But when s is not an integer, the various terms involving s in their index are multiform functions, and to ensure uniformity we have to assign definite cross-cuts to the logarithms which arise in the equivalent exponentials. When, as under the limitations for which the equality (A) has been established, these cross-cuts are formed by a line outside the smaller angle between the axes of ω_1 and ω_2, the expansion is perfectly valid; but when the common cross-cut lies within this angle, terms arise similar to those which occurred in Part II. of this paper, which are multiples of $2\pi\iota$, and involve n.





And, therefore, if we attempted in this case (see the second figure) to define $\zeta_2(s, a \mid \omega_1, \omega_2)$ as the absolute term in an asymptotic equality such as (A) § 38, where for complex values of s the principal value of each term is taken, we should ultimately

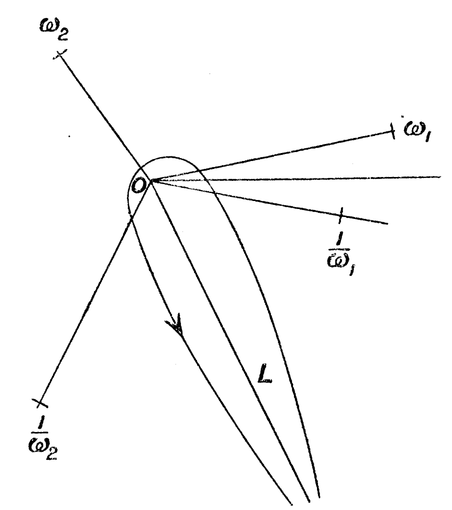
* POINCARÉ, 'Acta Mathematica,' vol. 8, pp. 295–344; 'Mécanique Céleste,' vol. 2, pp. 12–14.

find that $\zeta_2(s, a \mid \omega_1, \omega_2)$ as so defined would involve n. In other words, we should have made an assumption which could not be justified.

If we wish to obtain an expansion valid for all values of ω_1 and ω_2, we must consider as our starting point the integral

$$\frac{\iota\Gamma(-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})};$$

where $(-z)^{s-1} = e^{(s-1)\log(-z)}$, the logarithm being rendered uniform by a cut along an axis L, coinciding with the bisector of the smaller angle between the axes of $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$, and where the integral is taken along a contour having this axis L for axis (as in the figure), and enclosing the origin, but no other possible pole of the subject of integration. That value of $\log(-z)$ is to be taken which is such that the imaginary part of the initial value of $\log(-L)$ lies between 0 and $-2\pi\iota$.



This integral of course is only valid when a lies between the smaller area bounded by the axes of ω_1 and ω_2, or, as we may say, when a is positive with respect to ω_1 and ω_2. We notice that the line L is uniquely defined, since the ratio ω_2/ω_1 cannot be real and negative. The definition of the integral is not complete when ω_1 and ω_2 include and are equally inclined to the axis of -1; in this case we may take L to be a line nearly coinciding with this axis.

We now define the double Riemann ζ function, when the variable a is positive with respect to the ω's, and s, ω_1, and ω_2 have any complex values, by means of the equality

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(1-s)}{2\pi} e^{2M s\pi i}\int_L \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

where $M = 0$, unless the axes of $\frac{1}{L}$ and $(\omega_1 + \omega_2)$ include the axis of -1, in which case

$M = \mp 1$, the upper or lower sign being taken as $I(\omega_1 + \omega_2)$ is positive or negative.

§ 40. Let us take now the integral which has just been defined, and apply to it the procedure of § 38.

We shall evidently have to consider integrals of the type

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-nz}(-z)^{s-1}dz,$$

where the axis of n lies within the smaller angle between the axes of ω_1 and ω_2. We can at once see that this integral

$$= \frac{\iota\Gamma(1-s)}{2\pi} e^{2\mu s\pi\iota}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz$$

where $\mu = 0$, unless the axes of L and $\frac{1}{n}$ embrace the axis of -1, in which case

$\mu = \pm 1$, the upper or lower sign being taken as $I(n)$ is positive or negative.

Let $n = re^{\iota\theta}$, $\omega_1 = ae^{\iota\alpha}$, $\omega_2 = be^{\iota\beta}$, where θ, α and β are measured between 0 and 2π by rotation in the positive direction from the positive half of the real axis. The axis of L proceeds from the origin to the point $e^{-\frac{1}{2}(\alpha+\beta)}$ and therefore where z is at a distance ρ along the axis of L,

$$nz = r\rho\, e^{\iota\left(\theta - \frac{\alpha+\beta}{2}\right)}.$$

This quantity has its real part positive when

$$-\frac{\pi}{2} < \theta - \frac{\alpha+\beta}{2} < \frac{\pi}{2},$$

a relation which is satisfied when θ lies between α and β, and the difference between α and β is less than π. It is also satisfied when the axis of L proceeds from the origin to the point $e^{-\iota\left(\frac{\alpha+\beta}{2}+\epsilon\right)}$; where ϵ is a quantity less than half the excess of π over $\alpha \backsim \beta$. We see then that the axis of $\frac{1}{n}$ lies within a range of a right angle on either side of the axis of L. Hence by the propositions previously proved [" Theory of the Gamma Function," §§ 33, 34],

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-n}(-z)^{s-1}dz = \frac{\iota\Gamma(1-s)}{2\pi}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz,$$

unless the axes of L and $\frac{1}{n}$ embrace the negative half of the real axis. In this latter case, since the imaginary part of the initial value of $\log(-L)$ lies between 0 and $-2\pi\iota$, as also does that of $\log\left(-\frac{1}{n}\right)$, we are giving a different prescription to the many-valued function which occurs in the subject of integration.

We therefore have

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-nz}(-z)^{s-1}dz = \frac{\iota\Gamma(1-s)}{2\pi}\, e^{2\mu s\pi\iota}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz$$

where μ has the values which have been assigned to it.

Now $$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz = e^{s\log\left(n^{-1}\right)},$$

the logarithm having its principal value ("Theory of the Gamma Function," p. 107).

Hence $$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-nz}(-z)^{s-1}dz = e^{-s[\log n + 2\mu'\pi\iota]},$$

where $\mu' = 0$, unless n and $\frac{1}{L}$ embrace the axis of -1, in which case $\mu' = \pm 1$, the upper or lower sign being taken as $I\left(\frac{1}{L}\right)$ is positive or negative.

Finally then

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-nz}(-z)^{s-1}dz = \frac{1}{n^s},$$

where the latter function $= e^{-s\log n}$, where $\log n$ has a cross-cut along the axis of $-\frac{1}{L}$, and is real when n is real and positive. In other words, $\log n$ has its principal value with respect to the axis of $-\frac{1}{L}$.

§ 41. If now we apply to the integral

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$

the procedure of § 38, we shall, for all values of s, a, ω_1, and ω_2, such that a is positive with respect to the ω's, obtain the asymptotic equality

$$\sum_{\omega_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s} = \frac{\iota\Gamma(1-s)}{2\pi}\int_L \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$
$$+\frac{1}{n^{s-2}(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(p\omega_1+q\omega_2)^{s-2}}-\frac{1}{(p\omega_1)^{s-2}}-\frac{1}{(q\omega_2)^{s-2}}\right\}$$
$$-\frac{2a+\omega_1+\omega_2}{2(s-1)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}-\frac{1}{(pn\omega_1)^{s-1}}-\frac{1}{(qn\omega_2)^{s-1}}\right\}$$

[OVER]

$$- \frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}} + \frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\} + \frac{{}_2S'_1(a+\omega_1+\omega_2)}{(pn\omega_1+qn\omega_2)^s} - \frac{{}_2S'_1(a+\omega_1)}{(pn\omega_1)^s} - \frac{{}_2S'_1(a+\omega_2)}{(qn\omega_2)^s}$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+s}}\binom{m+s-1}{m}\left\{\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(p\omega_1+q\omega_2)^{m+s}} - \frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(p\omega_1)^{m+s}} - \frac{{}_2S_m(a+\omega_2)+{}_2B_{m+1}}{(q\omega_2)^{m+s}}\right\},$$

wherein all the many-valued functions with s as index have their principal values with respect to the axis of $-\frac{1}{L}$. It proves convenient to consider these functions as having their principal value with respect to the axis of $-(\omega_1+\omega_2)$. In order that this may be the case, we must multiply the integral by

$$e^{2M s\pi i}$$

where, as in § 39, $\mu = 0$, unless $(\omega_1+\omega_2)$ and $\frac{1}{L}$ includes the axis of -1, in which case $M = \pm 1$, as $I(\omega_1+\omega_2)$ is negative or positive.

Remembering the definition of $\zeta_2(s, a \mid \omega_1, \omega_2)$ given at the end of § 39, we see that we obtain for our fundamental asymptotic equality an expression which in form is identical with (A) § 38, but in which the many-valued functions with s as index have their principal values with respect to the axis of $-(\omega_1+\omega_2)$. It is evident that the equality will hold for all values of a, and will thus serve to define $\zeta_2(s, a \mid \omega_1, \omega_2)$ for all values of s, a, ω_1, and ω_2.

§ 42. We proceed now to take such particular cases of the general asymptotic equality which has just been obtained as lead to expressions for the logarithm of the double gamma function and its derivatives.

Suppose that s is a positive integer greater than 2; then, making n infinite in the general asymptotic equality, we see that

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{(-)^s}{(s-1)!}\psi_2^{(s)}(a \mid \omega_1, \omega_2),$$

where

$$\psi_2^{(s)}(a \mid \omega_1, \omega_2) = \frac{d^s}{da^s}\log \Gamma_2(a \mid \omega_1, \omega_2).$$

This relation is true for all values of a, ω_1 and ω_2; it is the first of a series connecting the double zeta and double gamma functions.

Let us next put $s+\epsilon$ for s, where ϵ is a small real quantity and s is, as before, a positive integer greater than 2. Then, provided a is positive with respect to ω_1 and ω_2,

$$\zeta_2(s+\epsilon, a \mid \omega_1, \omega)_2 = \frac{\iota\Gamma(1-s-\epsilon)}{2\pi} e^{2M\pi\iota(s+\epsilon)} \int_L \frac{e^{-az}(-z)^{s-1+\epsilon}\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$

the integral being taken along the L-contour, and M being the integer defined at the end of § 39.

Hence if $\log(-z)$ is real when z is negative, and is rendered uniform by a cut along the positive direction of the axis L,

$$\begin{aligned}\zeta_2(s+\epsilon, a\,|\,\omega_1, \omega_2) &= -\frac{\iota}{2\pi}\cdot\frac{(-)^{s-1}}{(s-1)!}\cdot\frac{1}{\epsilon}\int_L dz \frac{e^{-az}(-z)^{s-1}e^{2\mathrm{M}s\pi\iota}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\left\{1+\gamma\epsilon+\ldots\right\} \\ &\times\left\{1-\epsilon\left(\frac{1}{1}+\ldots+\frac{1}{s-1}\right)-\ldots\right\}\left\{1+\epsilon\log(-z)+\ldots\right\} \\ &\times\left\{1+2\mathrm{M}\pi\iota\epsilon+\ldots\right\},\end{aligned}$$

for $$\Gamma(\epsilon-s+1)=\frac{\Gamma(\epsilon)}{(\epsilon-s+1)(\epsilon-s+2)\ldots(\epsilon-1)}$$

$$=\frac{(-)^{s-1}}{(s-1)!\,\epsilon}\left\{1-\gamma\epsilon+\ldots\right\}\left\{1+\epsilon\left(\frac{1}{1}+\ldots+\frac{1}{s-1}\right)+\ldots\right\}.$$

Now when s is an integer greater than 2,

$$\int_L\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}=0,$$

for the integral may be reduced to two line integrals which destroy one another, and an integral round a small circle enclosing the origin whose value is zero.

We have then, on making $\epsilon=0$,

$$\zeta(s, a\,|\,\omega_1, \omega_2)=\frac{\iota}{2\pi}\frac{(-)^{s-1}}{(s-1)!}\int_L\frac{e^{-az}(-z)^{s-1}\left\{-\log(-z)+2\mathrm{M}\pi\iota+\frac{1}{1}+\ldots+\frac{1}{s-1}-\gamma\right\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz.$$

But, when s is an integer greater than 2,

$$\int_L\frac{e^{-az}(-z)^{s-1}\left\{2\mathrm{M}\pi\iota+\frac{1}{1}+\ldots+\frac{1}{s-1}-\gamma\right\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz$$

vanishes for the reason just assigned.

We see, then, that when a is positive with respect to ω_1 and ω_2, and s is a positive integer greater than 2,

$$\zeta_2(s, a\,|\,\omega_1, \omega_2)=(-)^s\frac{\iota}{2\pi}\frac{1}{(s-1)!}\int_L\frac{e^{-az}(-z)^{s-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz,$$

and therefore under the same conditions,

$$\psi_2^{(s)}(a\,|\,\omega_2, \omega_2)=\frac{\iota}{2\pi}\int_L\frac{e^{-az}(-z)^{s-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz.$$

§ 43. Put now in this result $s=3$; then with the assigned limitations

$$\psi_2^{(3)}(a\,|\,\omega_1, \omega_2)=\frac{\iota}{2\pi}\int_L\frac{e^{-az}(-z)^2\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz,$$

and it is obvious that we may replace $\log(-z)$ in the subject of integration by $\log(-z)+\gamma$ without altering the value of the integral. Integrate successively with respect to a, and we obtain

$$\log \Gamma_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz + (1, a)^2,$$

where the coefficients of the additive quadratic form are constants with respect to a.

Remember that

$$\operatorname*{Lt}_{a=0}\left[a\Gamma_2(a \mid \omega_1, \omega_2)\right] = 1,$$

then we evidently have

$$\log \Gamma_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz + \lambda_1 a + \lambda_2 a^2$$
$$- \operatorname*{Lt}_{a=0}\left[\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{\omega_1 z})(1-e^{-\omega_2 z})} + \log a\right].$$

We now define the third double gamma modular form $\rho_2(\omega_1, \omega_2)$ by the relation

$$\log \rho_2(\omega_1, \omega_2) = -2M\pi\iota\, {}_2S'_1(o \mid \omega_1, \omega_2)$$
$$- \operatorname*{Lt}_{a=0}\left[\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz + \log a\right],$$

and we proceed to show that the constants λ_1 and λ_2 are such that

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2S_0(a)(M + m + m')2\pi\iota + {}_2S'_1(o)2M\pi\iota$$
$$+ \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

where the numbers m and m' have the values assigned in Part II.

If this relation is true we shall have

$$\log \frac{\Gamma_2^{-1}(a+\omega_1)}{\Gamma_2^{-1}(a)} = -S'_1(a \mid \omega_2)(M + m + m')2\pi\iota + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\,dz.$$

§ 44. Let us now consider this integral.

It is to be taken along a contour embracing the axis L, which we take to be the bisector of the smaller angle between the axes of $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$, unless such bisector should be the axis of -1, in which case we take it to be nearly coincident with this line. And in the subject of integration that value of $\log(-z)$ is to be taken which is real when z is real and negative, and is limited by a cross-cut along the axis of L.

Let us consider the relation of this integral to the integral

$$\frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_2 z})}\,dz,$$

which is defined in the same way with reference to the axis of $1/\omega_2$.

We proceed to show that

$$\frac{\iota}{2\pi}\int_{\mathrm{L}} = \frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} - 2\pi i\,\mu'\,\mathrm{S}'_1(a\,|\,\omega_2),$$

where $\mu' = 0$, unless the axes L and $1/\omega_2$ do not enclose the axis of -1, in which case

$$\mu' = \pm 1, \text{ according as } \mathrm{I}(\omega_2) \text{ is positive or negative.}$$

For take the integral along the contour embracing the axis L, and suppose the contour to expand so that it embraces also the axis of $1/\omega_2$.

Then since a is positive with respect to ω_1 and ω_2, and since the angle between the axes of L and $1/\omega_2$ is less than $\frac{1}{2}\pi$, the value of the integral will be unaltered, for its value along the part of the great circle at infinity between the axes of L and $1/\omega_2$ is zero.

Suppose now that the contour is taken to lie on the infinite-sheeted Neumann sphere, whose sheets intersect in the cross-cut from 0 to ∞, on which the subject of integration of the integral is uniform. We may, without altering the value of the integral, deform the cross-cut so as to take up a position along the axis of $1/\omega_2$, instead of along the axis L, provided that in doing so we do not give a new specification to the logarithm. The latter phenomenon will occur when $1/\omega_2$ and L embrace the axis of -1, in which case we take the first contour in a sheet in which z can assume real values, while the second is taken in one in which $\log(-z)$ for real negative values of z is equal to $\pm 2\pi\iota$.

After deformation of the cross-cut we may compress the contour so that it embraces the axis of $1/\omega_2$. It is easy to see by this repetition of the argument previously employed in the "Theory of the Gamma Function," that we have

$$\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\,dz$$

$$=\frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+2\mu'\pi\iota+\gamma\}}{1-e^{-\omega_2 z}}\,dz$$

$$=\frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\,dz - 2\mu'\pi\iota\,\mathrm{S}'_1(a\,|\,\omega_2),$$

where μ' has the value previously given.

§ 45. The assumption made in § 43 for the values of the constants λ_1 and λ_2 will therefore lead to the relation

$$\log\frac{\Gamma_2^{-1}(a+\omega_1)}{\Gamma_2^{-1}(a)} = \log\frac{\Gamma_1(a\,|\,\omega_2)}{\rho_1(\omega_2)}\,\mathrm{S}'_1(a\,|\,\omega_2)\{\mathrm{M}+m+m'+\mu'\}\,2\pi\iota,$$

for ("Theory of the Gamma Function," § 37)

$$\log\frac{\Gamma_1(a\,|\,\omega)}{\rho_1(\omega)} = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega z}}\,dz.$$

Now, by considering the various cases which can arise, it may be readily seen that

$$M + m' + \mu' = 0.$$

These constants all vanish unless ω_1 and ω_2 embrace the axis of -1. When this take place, suppose that ω_1 lies above, and ω_2 below, the real axis.

Then $\left.\begin{array}{l}\mu' = 0\\ m' = 1\\ M = 1\end{array}\right\}$ when $(\omega_1 + \omega_2)$ lies above and $1/L$ below the real axis.

$\left.\begin{array}{l}\mu' = 0\\ m' = 0\\ M = 0\end{array}\right\}$ when $(\omega_1 + \omega_2)$ and $1/L$ both lie below the real axis.

$\left.\begin{array}{l}\mu' = -1\\ m' = 1\\ M = 0\end{array}\right\}$ when $(\omega_1 + \omega_2)$ and $1/L$ both lie above the real axis.

$\left.\begin{array}{l}\mu' = -1\\ m' = 0\\ M = 1\end{array}\right\}$ when $(\omega_1 + \omega_2)$ lies below and $1/L$ above the real axis.

We get similar sets of values when the imaginary parts of ω_1 and ω_2 have opposite signs to those just assumed.

In all cases

$$M + m' + \mu' = 0,$$

and, therefore, with the values assumed for λ_1 and λ_2,

$$\frac{\Gamma_2^{-1}(a + \omega_1)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota S_1'(a \mid \omega_2)}.$$

Similarly we should find

$$\frac{\Gamma_2^{-1}(a + \omega_2)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_1)}{\rho_1(\omega_1)} e^{-Sm'\pi\iota S_1'(a \mid \omega_1)}.$$

But these are identically the fundamental formulæ for the double gamma function found in § 23.

The values assumed for λ_1 and λ_2 are therefore correct.

We have, then, the two important formulæ

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2S_0(a)(M + m + m')\, 2\pi\iota + {}_2S'_1(o)\, 2M\pi\iota$$
$$+ \frac{\iota}{2\pi} \int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz,$$

and $\log \rho_2(\omega_1, \omega_2) = -2M\pi\iota\, {}_2S'_1(o \mid \omega_1, \omega_2)$

$$- \underset{=0}{\mathrm{Lt}} \left[\frac{\iota}{2\pi} \int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz + \log a \right],$$

which express as contour integrals the double gamma function, and the third double gamma modular form. The first relation only holds when a is positive with respect to the ω's. The second is valid for all values of ω_1 and ω_2, subject of course to the dominant condition that ω_2/ω_1 is not real and negative.

It is worth noticing that the first formulæ may also be written

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2S_0(a)(m + m')\, 2\pi\iota + {}_2S'_1(a)\{2M\pi\iota + \gamma\} + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz.$$

§ 46. Subject to the condition that the real parts of a and L are positive, we may now express our contour-integrals as line-integrals.

Consider the integral

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz.$$

By hypothesis the logarithm has a cross-cut along the axis of L, the initial value of its imaginary part lying between 0 and $-2\pi\iota$. Hence if the contour of the integral be reduced to a straight line from ∞ to ϵ, where ϵ is a point on the axis of L very near the origin, a circle of small radius $|\epsilon|$ round the origin, and a straight line from ϵ back again to $+\infty$, we shall have

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z}(1-e^{-\omega_2 z})}\, dz = -\int_\epsilon^\infty (L) \frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-a\epsilon e^{\iota\theta}}\{\log\epsilon + \iota(\theta-\pi)\}}{(1-e^{-\omega_1\epsilon e^{\iota\theta}})(1-e^{-\omega_2\epsilon e^{\iota\theta}})}\, d\theta.$$

The logarithm in this second integral, which results from the small circular contour surrounding the origin, has its principal value. The integral itself is evidently equal to (§ 15)

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta[\log\epsilon + \iota(\theta-\pi)]\Big[\frac{e^{-2\iota\theta}}{\omega_1\omega_2\epsilon^2} - \frac{{}_2S_1^{(2)}(a)}{\epsilon}e^{-\iota\theta} + \frac{{}_2S'_1(a)}{1!} + \text{terms involving positive powers of } \epsilon\Big]$$

$$= {}_2S'_1(a)\log\epsilon - \frac{1}{2\omega_1\omega_2\epsilon^2} + \frac{{}_2S_1^{(2)}(a)}{\epsilon} + \text{terms which vanish with } |\epsilon|.$$

Thus

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz$$

$$= -\int_\epsilon^\infty (L)\frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + {}_2S'_1(a)\log\epsilon - \frac{1}{2\omega_1\omega_2\epsilon^2} + \frac{{}_2S_1^{(2)}(a)}{\epsilon} + \text{terms which vanish with } |\epsilon|.$$

Now, as has been seen in the "Theory of the Gamma Function," § 28 cor., when $\mathfrak{R}$ (L) is positive,

$$\int_{\epsilon}^{\infty}(\mathrm{L})\, e^{-z}\frac{dz}{z} = -\log\epsilon - \gamma + \text{terms which vanish with } |\epsilon|.$$

And evidently

$$\int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z^2} = \frac{1}{\epsilon}, \qquad \int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z^3} = \frac{1}{2\epsilon^2}.$$

The integral under consideration is thus equal to

$$\int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{\omega_1\omega_2 z^2} + \frac{{}_2S_1^{(2)}(a)}{z} - e^{-z}{}_2S'_1(a)\right\} - \gamma\, {}_2S'_1(a)$$
$$+ \text{ terms which vanish with } |\epsilon|.$$

If now we make ϵ coincide with the origin, the integral last written remains finite and we have

$$\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-1}\log(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{\omega_1\omega_2 z^2} + \frac{{}_2S_1^{(2)}(a)}{z} - e^{-z}{}_2S'_1(a)\right\} - \gamma\, {}_2S'_1(a).$$

This equality may equally be written

$$\log\frac{\Gamma_2(a\,|\,\omega_1,\omega_2)}{\rho_2(\omega_1,\omega_2)} - {}_2S_0(a)(\mathrm{M}+m+m')\,2\pi\iota + {}_2S'_1(o)\,2\mathrm{M}\pi\iota$$
$$= \frac{\iota}{2\pi}\int_{\mathrm{L}}^{\infty}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{{}_2S_1^{(3)}(a)}{z^2} + \frac{{}_2S_1^{(2)}(a)}{z} - e^{-z}{}_2S'_1(a)\right\}$$

under the assigned conditions that a is positive with respect to the ω's, and that the real part of L is positive. We thus express the logarithm of the double gamma function as a line-integral.

In order to obtain a line integral for $\log\rho_2(\omega_1,\omega_2)$, we notice that we have

$$\log a = \int_0^{\infty}(e^{-z} - e^{-az})\frac{dz}{z},$$

and therefore

$$\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + \log a$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - e^{-az} - \frac{{}_2S_1^{(3)}(a)}{z^2} + \frac{{}_2S_1^{(2)}(a)}{z} + e^{-z}\{1 - {}_2S'_1(a)\}\right\}$$

Therefore on making $a = 0$, we have by § 45,

$$\log \rho(\omega_1, \omega_2) = -2M\pi\iota\, {}_2B_1(\omega_1, \omega_2)$$
$$- \int_0^\infty (L) \frac{dz}{z} \left\{ \frac{1}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} - 1 - \frac{{}_2S_1^{(3)}(o)}{z^2} + \frac{{}_2S_1^{(2)}(o)}{z} + e^{-z}[1 - {}_2S'_1(a)] \right\}.$$

On differentiating the formulæ which express the logarithm of the double gamma function as line and contour integrals, we obtain

$$\psi'_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_L \frac{e^{-az} \log(-z)\, dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + {}_2S'_0(a) \Big[\gamma + (M + m + m') \,.\, \pi\iota \Big]$$
$$= \int_0^\infty (L) \frac{dz}{z} \left\{ \frac{-ze^{-az}}{(1 - e^{-\omega_2 z})(1 - e^{-\omega_2 z})} + \frac{{}_2S_0^{(2)}(a)}{z} - e^{-z}\, {}_2S'_0(a) \right\} + {}_2S'_0(a)[(M + m + m')2\pi\iota].$$

Similarly, again differentiating,

$$\psi_2^{(2)}(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_L \frac{e^{-az}(-z)\{\log(-z)\}dz}{-e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + {}_2S'^{(2)}_0(a) \Big[\gamma + 2\pi\iota(M + m + m' \Big]$$
$$= - \int_0^\infty (L) z dz \left\{ \frac{-e^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + \frac{e^{-z}}{z^2}\, {}_2S_0^{(2)}(a) \right\} + {}_2S_0^{(2)}(a) \Big[(M + m + m') \,.\, 2\pi\iota \Big].$$

And, if s be greater than 2,

$$\psi^{(s)}(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1} \log(-z)\, dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} = - \int_0^\infty (L) \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz.$$

Notice that, when we have the more narrow restrictions, the real parts of a, ω_1, and ω_2 are all positive, the constants m, m', and M all vanish, and there is a substantial simplification in the formulæ.

§ 47. We may now deduce expressions as line and contour integrals for the first and second double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2) \text{ and } \gamma_{22}(\omega_1, \omega_2).$$

We have seen (§ 22) that

$$-\psi_2^{(1)}(a \mid \omega_1, \omega_2) = \gamma_{22}(\omega_1, \omega_2) + a\gamma_{21}(\omega_1, \omega_2) + \frac{1}{a}$$
$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \left[\frac{1}{a + m_1\omega_1 + m_2\omega_2} - \frac{1}{m_1\omega_2 + m_2\omega_2} + \frac{a}{(m_1\omega_1 + m_2\omega_2)^2} \right],$$

and, therefore, on making $a = 0$,

$$\gamma_{22}(\omega_1, \omega_2) = - \left[\psi_2^{(1)}(a \mid \omega_1, \omega_2) + \frac{1}{a} \right]_{a=0},$$

so that, by the last paragraph,

$$\gamma_{22}(\omega_1, \omega_2) = \int_0^\infty (L)\, dz \left\{ \frac{1}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} - 1 - \frac{{}_2S_1^{(3)}(o)}{z^2} + \frac{e^{-z}}{z}\, {}_2S_1^{(2)}(o) \right\}$$
$$- {}_2S'_0(o)\, 2\pi\iota(M + m + m').$$

The additive term will of course vanish when the real parts of ω_1 and ω_2 are positive.

Similarly, we have

$$\gamma_{21}(\omega_1, \omega_2) = -\left[\psi_2^{(2)}(a \mid \omega_1, \omega_2) - \frac{1}{a^2}\right]_{a=0}$$

$$= \underset{a=0}{\mathrm{L}t}\left[-\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz + \frac{1}{a^2}\right] - {}_2\mathrm{S}_0^{(2)}(a)2\pi\iota(\mathrm{M}+m+m'),$$

so that the first double gamma modular form is expressed as a line integral by the formula

$$\gamma_{21}(\omega_1, \omega_2) = \int_0^\infty (-z)\,dz\left\{\frac{1}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - 1 - \frac{e^{-z}}{z^2}\,{}_2\mathrm{S}_1^{(3)}(o)\right\}$$
$$- {}_2\mathrm{S}_1^{(3)}(o)2\pi\iota(\mathrm{M}+m+m').$$

It will be noted that for the modular forms

$$\rho_2(\omega_1, \omega_2), \qquad \gamma_{21}(\omega_1, \omega_2), \qquad \gamma_{22}(\omega_1, \omega_2),$$

we have, by making a vanish, obtained line integrals which are in general finite, although in our fundamental formulæ the restriction was made that the real part of a should be positive.

This restriction was necessary to ensure that the contour integral should be finite at infinity. It is clear from the mode of generation of the line integrals, that the process which has been carried out is perfectly valid, since by the introduction of the terms $\log a$, $-\frac{1}{a^2}$, $\frac{1}{a}$ allowance has been made for the manner in which the contour integral tends to an infinite value as a tends to zero.

§ 48. At the beginning of § 43 we entered on the investigation which has just been given by integrating with respect to a under the sign of contour integration, and in this way we deduced the contour integral for $\log \Gamma_2(a)$ from that for $\psi_2^{(3)}(a)$.

We now proceed to show how the contour integral for $\log \Gamma_2(a)$ may be obtained without the employment of this process.

For this purpose we take the fundamental asymptotic equality of § 38, valid for all values of s, a, ω_1 and ω_2, the many-valued functions with s as index having their principal values with respect to the axis of $-(\omega_1 + \omega_2)$.

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn} \frac{1}{(a+m_1\omega_1+m_2\omega_2)^s} = \zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}} - \frac{1}{(pn\omega_1)^{s-2}} - \frac{1}{(qn\omega_2)^{s-2}}\right\}$$
$$- \frac{2a+\omega_1+\omega_2}{2(s-1)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}} - \frac{1}{(pn\omega_1)^{s-1}} - \frac{1}{(qn\omega_2)^{s-1}}\right\} - \frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}} + \frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}.$$

[OVER]

$$+\frac{{}_2S_1'(a+\omega_1+\omega_2)}{(pn\omega_1+qn\omega_2)^s}-\frac{{}_2S_1'(a+\omega_1)}{(pn\omega_1)^s}-\frac{{}_2S_1'(a+\omega_2)}{(qn\omega_2)^s}$$

$$+\sum_{m=1}^{\infty}\frac{(-)^m}{n^{m+s}}\binom{m+s-1}{m}\left\{\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(p\omega_1+q\omega_2)^{m+s}}-\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(p\omega_1)^{m+s}}-\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(q\omega_2)^{m+s}}\right\}$$

where, if a is positive with respect to the ω's, $\zeta(s, a \mid \omega_1, \omega_2)$ may be expressed by the integral

$$e^{2Ms\pi i}\cdot\frac{\iota}{2\pi}\Gamma(1-s)\cdot\int_L\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\quad(\S\ 39.)$$

Make now $s=\epsilon$, where ϵ is very small; expand the various terms of the asymptotic equality in powers of ϵ, neglecting those higher than the first, and we obtain, if the real part of a is positive,

$$(pn+1)(qn+1)-\epsilon\log\prod_{m_1=0}^{pn}\prod_{m_2=0}^{qn}(a+m_1\omega_1+m_2\omega_2)$$

$$=\frac{\iota}{2\pi}\int_L\frac{e^{-az}(-z)^{-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\{1+\epsilon\log(-z)\}\{1+\gamma\epsilon\}\{1+2M\epsilon\pi\iota\}\,dz$$

$$+\frac{1}{2\omega_1\omega_2}\{1+\epsilon(\tfrac{1}{1}+\tfrac{1}{2})\}\{n^2(p\omega_1+q\omega_2)^2[1-\epsilon\log(p\omega_1 n+q\omega_2 n)]$$
$$-(np\omega_1)^2[1-\epsilon\log pn\omega_1]-(nq\omega_2)^2[1-\epsilon\log qn\omega_2]\}$$

$$+{}_2S_1^{(2)}(a+\omega_1+\omega_2)(1+\epsilon)\Big[n(p\omega_1+q\omega_2)[1-\epsilon\log(pn\omega_1+qn\omega_2)]$$
$$-np\omega_1[1-\epsilon\log pn\omega_1]-nq\omega_2[1-\epsilon\log qn\omega_2]\Big]$$

$$+(1+\epsilon)[pn(1-\epsilon\log pn\omega_1)+qn(1-\epsilon\log qn\omega_2)]$$

$$+{}_2S_1'(a+\omega_1+\omega_2)[1-\epsilon\log(pn\omega_1+qn\omega_2)]-{}_2S_1'(a+\omega_1)[1-\epsilon\log pn\omega_1]$$
$$-{}_2S_1'(a+\omega_2)[1-\epsilon\log qn\omega_2]$$

$$+\sum_{m=1}^{\infty}\frac{(-)^m}{n^m}\frac{\epsilon}{m}\left[\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(p\omega_1+q\omega_2)^m}-\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(p\omega_1)^m}-\frac{{}_2S_m(a+\omega_2)+{}_2B_{m+2}}{(q\omega_1)^m}\right].$$

This equality will hold for all values of s, a, ω_1, ω_2 if the integral be replaced by $\zeta_2(\epsilon, a \mid \omega_1, \omega_2)$, the logarithms having their principal value with respect to $-(\omega_1+\omega_2)$.

Equate now the absolute terms in this asymptotic equality, and we find, if a is positive with respect to the ω's,

$$1-\frac{\iota}{2\pi}\int_L\frac{e^{-az}(-z)^{-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz+{}_2S_1'(a+\omega_1+\omega_2)-{}_2S_1'(a+\omega_1)-{}_2S_1'(a+\omega_2).$$

But we have seen (§ 17) that

$${}_2S_1'(a+\omega_1+\omega_2)-{}_2S_1'(a+\omega_1)-{}_2S_1'(a+\omega_2)=-{}_2S_1(a)+1.$$

Hence, provided a is positive with respect to the ω's,

$$ {}_2S_1'(a) = \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz, $$

and for all values of a, ω_1 and ω_2

$$ \zeta_2(o, a\,|\,\omega_1, \omega_2) = {}_2S_1'(a\,|\,\omega_1, \omega_2). $$

§ 49. Secondly, equate terms involving the first power of ϵ.

We obtain the asymptotic equality

$$ \begin{aligned} &-\log \prod_{m_1=0}^{pn}\prod_{m_2=0}^{qm}(a+m_1\omega_1+m_2\omega_2) \\ &= \left[\frac{\partial}{\partial s}\zeta_2(s, a\,|\,\omega_1, \omega_2)\right]_{s=0} + n^2(\tfrac{1}{1}+\tfrac{1}{2})pq - n^2\log n\,{}_2S_1^{(3)}(o)\,[p\omega_1+q\omega_2)^2 \\ &\qquad\qquad - (p\omega_1)^2 - (q\omega_2)^2] \\ &\quad - n^2{}_2S_1^{(3)}(o)\,\tfrac{1}{2}\,[(p\omega_1+q\omega_2)^2\log(p\omega_1+q\omega_2) - (p\omega_1)^2\log(p\omega_1) - (q\omega_2)^2\log q\omega_2] \\ &\quad + n(p+q) - n\log n\,[{}_2S_1^{(2)}(a+\omega_1+\omega_2)(p\omega_1+q\omega_2) - {}_2S_1^{(2)}(a+\omega_1)\,p\omega_1 \\ &\qquad\qquad - {}_2S_1^{(2)}(a+\omega_2)\,p\omega_2] \\ &\quad - n\,[{}_2S_1^{(2)}(a+\omega_1+\omega_2)(p\omega_1+q\omega_2)\log(p\omega_1+q\omega_2) - {}_2S_1^{(2)}(a+\omega_1)\,p\omega_1\log p\omega_1 \\ &\qquad\qquad - {}_2S_1^{(2)}(a+\omega_2)\,q\omega_2\log q\omega_2] \\ &\quad - [{}_2S_1'(a+\omega_1+\omega_2)\log(p\omega_1+q\omega_2) - {}_2S_1'(a+\omega_1)\log p\omega_1 \\ &\qquad\qquad - {}_2S_1'(a+\omega_2)\log q\omega_2] \\ &\quad - \log n\,[{}_2S_1'(a+\omega_1+\omega_2) - {}_2S_1'(a+\omega_1) - {}_2S_1'(a+\omega_2)] \\ &\quad + \sum_{m=1}^{\infty}\frac{(-)^m}{(m+1)\,mn^m}\left[\frac{{}_2S'_{m+1}(a+\omega_1+\omega_2)}{(p\omega_1+q\omega_2)^m} - \frac{{}_2S'_{m+1}(a+\omega_1)}{(p\omega_1)^m} - \frac{{}_2S'_{m+1}(a+\omega_2)}{(p\omega_2)^m}\right], \end{aligned} $$

valid for all values of s, a, ω_1 and ω_2, provided the logarithms have their principal values with respect to the axis of $-(\omega_1+\omega_2)$.

In order that the labour of writing down cumbrous formulæ like the one just obtained may be diminished as much as possible, we propose to introduce a symbolic notation suggested by CAYLEY's notation of matrices.*

If $f(z)$ be any function of z, we shall represent symbolically

$$ f(z+\omega_1+\omega_2) - f(z+\omega_1) - f(z+\omega_2) $$

by $F_2[f(z+\omega)]$, the suffix 2 denoting that we are dealing with two parameters.

Thus the difference equation for double Bernoullian numbers (§ 17) is written

$$ F_2[{}_2S_n(z+\omega)] = -{}_2S_n(z) + z^n. $$

Similarly $\qquad F_2[{}_2S_1^{(2)}(a+\omega)\,p\omega\log p\omega]$ denotes the function

* CAYLEY, 'Collected Works,' vol. 2. The corresponding theory for multiple gamma functions will be developed by employing a symbolic notation *ab initio*.

$${}_2S_1^{(2)}(a+\omega_1+\omega_2)(p\omega_1+q\omega_2)\log(p\omega_1+q\omega_2) - {}_2S_1^{(2)}(a+\omega_1)p\omega_1\log p\omega_1 - {}_2S_1^{(2)}(a+\omega_2)q\omega_2\log q\omega_2.$$

[The analogy with the matrix notation would be more complete if p and q were replaced by p_1 and p_2. The convention adopted here is, however, quite natural.]

And now our asymptotic equality may be written

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a+m_1\omega_1+m_2\omega_2)$$

$$= -\left[\frac{\partial}{\partial s}\zeta_2(s, a \mid \omega_1, \omega_2)\right]_{s=0} + pq\left[n^2\log n - n^2\left(\tfrac{1}{1}+\tfrac{1}{2}\right)\right] + (p+q)\left[n\log n - n\right]$$

$$+ \frac{n^2}{2}F_2\left[{}_2S_1^{(3)}(\omega)(p\omega)^2\log(p\omega)\right] + nF_2\left[{}_2S_1^{(2)}(a+\omega)(p\omega)\log(p\omega)\right]$$

$$+ F_2\left[{}_2S_1'(a+\omega)\log(p\omega)\right] + \log n\left[1 - {}_2S_1'(a)\right]$$

$$+ \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{mn^m}F_2\left\{\frac{{}_2S_m(a+\omega)+{}_2B_{m+1}}{(p\omega)^m}\right\},$$

for, as may be readily proved,

$${}_2S_1^{(2)}(a+\omega_1+\omega_2)(p\omega_1+q\omega_2) - {}_2S_1^{(2)}(a+\omega_1)p\omega_1 - {}_2S_1^{(2)}(a+\omega_2)p\omega_2 = p+q.$$

When the variable a is positive with respect to the ω's, we note that the part of the absolute term in this asymptotic equality which is equal to

$$-\left[\frac{\partial}{\partial s}\zeta_2(s, a \mid \omega_1, \omega_2)\right]_{s=0}$$

may be written

$$\frac{-\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\gamma+\log(-z)\}\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - 2M\pi\iota\,{}_2S_1'(a),$$

which is the expression which has been proved equal to ${}_2S_0(a)\ 2(m+m')\pi\iota - \log\dfrac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}$ by the process of differentiation under the sign of contour integration.

§ 50. But if we take the expression for $\log\Gamma_2(a)$ which has been obtained in § 24,

$$-\log\Gamma_2(a) = \frac{a^2}{2}\gamma_{21}(\omega_1, \omega_2) + a\gamma_{22}(\omega_1, \omega_2) + \log a + \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}{}' \left[\log(a+\Omega) - \log\Omega - \frac{a}{\Omega} + \frac{a^2}{2\Omega^2}\right],$$

and write it

$$-\log\Gamma_2(a) = \frac{a^2}{2}\gamma_{21}(\omega_1, \omega_2) + a\gamma_{22}(\omega_1, \omega_2) + \log a + \operatorname*{Lt}_{n=\infty}\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}{}' \left[\log(a+\Omega) - \log\Omega - \frac{a}{\Omega} + \frac{a^2}{2\Omega^2}\right],$$

we may obtain this expression independently.

For putting $a = 0$ in the asymptotic expansion of

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a + \Omega)$$

we find

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (m_1\omega_1 + m_2\omega_2)$$

$$= - \underset{\substack{a=0\\=0}}{\mathrm{L}t} \left[\frac{\partial}{\partial s} \zeta_2 (s, a \,|\, \omega_1, \omega_2) + \log a\right] + pq\left[n^2 \log n - n^2 \left(\tfrac{1}{1} + \tfrac{1}{2}\right)\right.$$

$$+ (p + q)[n \log n - n] + n^2 \mathrm{F}_2 [{}_2\mathrm{S}_1^{(3)} (\omega)\, p\omega \log p\omega] + n\mathrm{F}_2 [{}_2\mathrm{S}_1^{(2)} (\omega)\, p\omega \log p\omega]$$

$$+ \mathrm{F}_2 [{}_2\mathrm{S}_1' (\omega) \log p\omega] + \log n [1 - {}_2\mathrm{S}_1' (0)]$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} \mathrm{F}_2 \left\{\frac{{}_2\mathrm{S}_m (\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^m}\right\} \quad . \quad . \quad . \quad . \quad (1).$$

This is the extension of STIRLING'S Theorem to two parameters. If for all values of ω_1 and ω_2, we put

$$\log \rho_2 (\omega_1, \omega_2) = - \underset{\substack{a=0\\s=0}}{\mathrm{L}t} \left[\frac{\partial}{\partial s} \zeta (s, a \,|\, \omega_1, \omega_2) + \log a\right],$$

we may call $\rho_2 (\omega_1, \omega_2)$ the double Stirling function of the parameters ω_1 and ω_2. It is the same as the third double gamma modular form previously defined. For we have by § 43,

$$\log \rho_2 (\omega_1, \omega_2) = \underset{a=0}{\mathrm{L}t} \left[\frac{\iota}{2\pi} \int \frac{e^{-az} (-z)^{-1} \{\log (-z) + \gamma\} dz}{(1 - e^{-\omega_1 z}) (1 - e^{-\omega_2 z})} - \log a\right] - 2\mathrm{M}\pi\iota {}_2\mathrm{S}_1' (0 \,|\, \omega_1, \omega_2) .$$

$$= \underset{\substack{a=0\\s=0}}{\mathrm{L}t} \left[-\frac{\partial}{\partial s} \zeta (s, a \,|\, \omega_1, \omega_2) - \log a\right] \quad \text{by § 42.}$$

We now see the exact analogy between the function $\rho_2 (\omega_1, \omega_2)$ and the simple Stirling function $\rho_1 (\omega) = \sqrt{(2\pi/\omega)}$ as defined in § 31 of the "Theory of the Gamma Function."

For a brief inspection shows us that the result of § 30 of that theory may be written

$$\log \prod_{m_1=1}^{pn} (m_1 \omega) = p (n \log n - n) + n\{\mathrm{S}_1^{(2)} (\omega)\, p\omega \log p\omega\}$$

$$+ [1 + \mathrm{S}'_1 (o)] \log n + \log \rho_1 (\omega) + \mathrm{S}_1' (\omega) \log p\omega$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} \frac{\mathrm{S}_m (\omega) + {}_1\mathrm{B}_{m+1}}{(p\omega)^m}$$

which is the complete form of STIRLING'S Theorem for a single parameter.

The analogy between this asymptotic expansion and

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (m_1\omega_1 + m_2\omega_2) = pq\left[n^2 \log n - n^2\left(\tfrac{1}{1} + \tfrac{1}{2}\right)\right] + n^2 F_2\left[{}_2S_1^{(3)}(\omega)\, p\omega \log p\omega\right]$$
$$+ (p+q)\left[n \log n - n\right] + nF_2\left[{}_2S_1^{(2)}(\omega)\, p\omega \log p\omega\right]$$
$$+ \left[1 - {}_2S'_1(o)\right]\log n + \log \rho_2(\omega_1, \omega_2) + F_2\left[{}_2S'_1(\omega) \log p\omega\right]$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} F_2\left\{\frac{{}_2S_m(\omega) + {}_2B_{m+1}}{(p\omega)^m}\right\}$$

is so evident as to determine the nomenclature.

Note in the second place that the fundamental asymptotic expansion (A) of § 38 may, with the symbolic notation subsequently introduced, be written

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s} = \zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{(s-1)(s-2) \,.\, n^{s-2}} F_2\left\{\frac{{}_2S_1^{(3)}(\omega)}{(p\omega)^{s-2}}\right\}$$
$$- \frac{1}{(s-1) \,.\, n^s} F_2\left\{\frac{{}_2S_1^{(2)}(a+\omega)}{(p\omega)^{s-1}}\right\} + \frac{1}{n^s} F_2\left\{\frac{{}_2S'_1(a+\omega)}{(p\omega)^s}\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+s}} \binom{m+s-1}{m} F_2\left\{\frac{{}_2S_m(a+\omega) + {}_2B_{m+1}}{(p\omega)^{m+s}}\right\}$$

§ 51. If in this equality, true for all values of s, a, ω_1 and ω_2, we make $s = 1$, we obtain, since

$$F_2\left\{{}_2S_1^{(3)}(\omega)(p\omega)\right\} = 0$$
$$F_2\left\{{}_2S_1^{(2)}(a+\omega)\right\} = -\,{}_2S_1^{(2)}(a) \;.$$

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{a + m_1\omega_1 + m_2\omega_2} = \operatorname*{Lt}_{s=1}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1}\right] + \frac{1}{n} F_2\left\{\frac{{}_2S'_1(a+\omega)}{p\omega}\right\}$$
$$+ nF_2\left\{{}_2S_1^{(3)}(\omega)\, p\omega \log p\omega\right\} - {}_2S_1^{(2)}(a) \log n + F_2\left\{{}_2S_1^{(2)}(a+\omega) \log p\omega\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+1}} F_2\left\{\frac{{}_2S_m(a+\omega) + {}_2B_{m+1}}{(p\omega)^{m+1}}\right\},$$

which is equivalent to the result of differentiating the asymptotic expansion of § 49.

If now we make $a = 0$, we obtain

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn}{}' \frac{1}{m_1\omega_1 + m_2\omega_2} = \operatorname*{Lt}_{\substack{a=0\\ s=1}}\left[\zeta(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1} - \frac{1}{a}\right]$$
$$+ nF_2\left\{{}_2S_1^{(3)}(\omega) p\omega \log p\omega\right\} - {}_2S_1^{(2)}(o) \log n + F_2\left\{{}_2S_1^{(2)}(\omega) \log p\omega\right\}$$
$$+ \sum_{m=0}^{\infty} \frac{(-)^m}{n^{m+1}} F_2\left\{\frac{{}_2S_m(\omega) + {}_2B_{m+1}}{(p\omega)^{m+1}}\right\}.$$

We now see that

$$\operatorname*{Lt}_{n=\infty}\left[\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn}{}' \frac{1}{m_1\omega_1 + m_2\omega_2} + {}_2S_1^{(2)}(o) \log n - F\left\{{}_2S_1^{(2)}(\omega) \log p\omega\right\} - nF_2\left\{{}_2S_1^{(3)}(\omega)\, p\omega \log p\omega\right\}\right]$$
$$= \operatorname*{Lt}_{\substack{a=0\\ =0}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1} - \frac{1}{a}\right],$$

which is a quantity independent of p and q.

But when $p = q = 1$, the expression on the left-hand side may be written

$$\underset{n=\infty}{\mathrm{Lt}}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\log n + \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\left\{\log(\omega_1+\omega_2) - \log\omega_1 - \log\omega_2\right\}\right.$$
$$\left. - (n+1)\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\log(\omega_1+\omega_2) + \frac{n+1}{\omega_1}\log\omega_2 + \frac{n+1}{\omega_2}\log\omega_1\right].$$

Since the principal values of the logarithms with respect to the axis of $-(\omega_1+\omega_2)$ are in all cases to be taken, we see from § 23 that this expression is equal to

$$\gamma_{22}(\omega_1, \omega_2) + {}_2S_1^{(2)}(o)\, 2(m+m')\pi\iota.$$

For denoting by a capital letter the logarithm which has its principal value with respect to the axis of -1, we have

$$\log(\omega_1+\omega_2) = \mathrm{Log}(\omega_1+\omega_2),$$
$$\log\omega_1 = \mathrm{Log}\,\omega_1 + 2m\pi\iota,$$
$$\log\omega_2 = \mathrm{Log}\,\omega_2 + 2m'\pi\iota.$$

And therefore

$$\underset{\substack{a=0\\s=1}}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1} - \frac{1}{a}\right]$$
$$= \underset{n=\infty}{\mathrm{Lt}}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\log n + \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\left\{\log(\omega_1+\omega_2) - \log\omega_1 - \log\omega_2\right\}\right.$$
$$- (n+1)\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\log(\omega_1+\omega_2) + \frac{n+1}{\omega_1}\log\omega_2 + \frac{n+1}{\omega_2}\log\omega_1$$
$$\left. + \frac{n+1}{\omega_1}2m\pi\iota + \frac{n+1}{\omega_2}2m'\pi\iota\right] - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}2(m+m')\pi\pi i$$
$$= \gamma_{21}(\omega_1, \omega_2) + {}_2S_1^{(2)}(o)\, 2(m+m')\pi\iota.$$

We notice that this formula agrees with our previous results. For by § 47

$$\gamma_{21}(\omega_1, \omega_2) + {}_2S'_1(o)[(m+m')2\pi\iota]$$
$$= -\underset{a=0}{\mathrm{Lt}}\left[\frac{\iota}{2\pi}\int_L \frac{e^{-az}\{\log(-z)+\gamma\}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + \frac{1}{a}\right] - 2M\pi\iota\,{}_2S_1(o),$$

and by § 42 this last expression

$$= \underset{\substack{a=0\\s=1}}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1} - \frac{1}{a}\right].$$

It is worth noticing that incidentally $\gamma_{22}(\omega_1, \omega_2)$ has been obtained as a limit in the more general form

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}}\left[\sum_0^{pn}\sum_0^{qn}\frac{1}{\Omega} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\log n\right.$$
$$+ \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\{\log(p\omega_1+q\omega_2) - \log p\omega_1 - \log q\omega_2\}$$

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$$-\left(\frac{pn+1}{\omega_2}+\frac{qn+1}{\omega_1}\right)\log(p\omega_1+q\omega_2)+\frac{pn+1}{\omega_2}(\log q\omega_1+2m\pi\iota)$$
$$+\frac{qn+1}{\omega_1}(\log q\omega_2+2m'\pi\iota)\Bigg],$$

where $\log(p\omega_1+q\omega_2)$ has its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

§ 52. Let us next put $s=2$ in the final equality of § 50.

We shall obtain, in the limit when n is infinite,

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^2}=\underset{s=2}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2S_1^{(3)}(a)}{s-2}\right]+(1+\log n)\,{}_2S_1^{(3)}(o)$$
$$-\ F_2\{{}_2S_1^{(3)}(\omega)\log p\omega\},$$

the logarithms having their principal values with respect to the axis of $-(\omega_1+\omega_2)$.

Thus $$\underset{\substack{s=2\\a=0}}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2S_1^{(3)}(0)}{s-2}-\frac{1}{a^2}\right]+\frac{1}{\omega_1\omega_2}$$
$$=\sum_0^{pn}\sum_0^{qn}{}'\frac{1}{(m_1\omega_1+m_2\omega_2)^2}-\frac{1}{\omega_1\omega_2}[\log n-\log(p\omega_1+q\omega_2)+\log p\omega_1+\log q\omega_2].$$

Now the left-hand side of this relation is independent of p and q, and therefore we see, by § 22, on putting $p=q=1$, that

$$\underset{\substack{s=2\\a=0}}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2S_1^{(3)}(o)}{s-2}-\frac{1}{a^2}\right]+\frac{1}{\omega_1\omega_2}=-\gamma_{21}(\omega_1,\omega_2)-2(m+m')\pi\iota\,{}_2S_1^{(3)}(o).$$

This formula again agrees with results which can be deduced from the integral formulæ. Incidentally $\gamma_{21}(\omega_1,\omega_2)$ has been expressed as the more general limit

$$\gamma_{21}(\omega_1,\omega_2)=\frac{1}{\omega_1\omega_2}\log n-\sum_0^{pn}\sum_0^{qn}\frac{1}{(m_1\omega_1+m_2\omega_2)^2}$$
$$-\frac{1}{\omega_1\omega_2}[\log(p\omega_1+q\omega_2)-\log p\omega_1-\log q\omega_2],$$

where $\log(p\omega_1+q\omega_2)$ has its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

§ 53. We can now finish the investigation indicated at the beginning of § 50, and obtain the expression for $\log\dfrac{\Gamma_2(a)}{\rho_2(\omega_1,\omega_2)}$ without integrating under the sign of contour integration. We have

$$-\log\Gamma_2(a\,|\,\omega_1,\omega_2)=\frac{a^2}{2}\gamma_{21}+a\gamma_{22}+\log a$$
$$+\underset{n=\infty}{\mathrm{L}t}\sum_{\substack{0\\m_1}}^{pn}\sum_{\substack{0\\m_2}}^{qn}{}'\left[\log(a+\Omega)-\log\Omega-\frac{a}{\Omega}+\frac{a^2}{2\Omega^2}\right],$$

where $\Omega=m_1\omega_1+m_2\omega_2$.

Utilising the results of §§ 49–52, we find, when a is positive with respect to the ω's,

$$-\log \Gamma_2(a) = \underset{n=\infty}{Lt}\left\{\frac{-\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz - 2M\pi\iota\,{}_2S_1'(a)\right.$$
$$+ nF_2\{[{}_2S_1^{(2)}(a+\omega) - {}_2S_1^{(2)}(\omega)]p\omega\log p\omega\} + F_2\{{}_2S_0(a+\omega)\log p\omega\}$$
$$+ \log n\{{}_2S_1'(o) - {}_2S_1'(a)\} - \log \rho_2(\omega_1, \omega_2)$$
$$+ a[-{}_2S_1^{(2)}(o)\,2(m+m')\pi\iota + {}_2S_1^{(2)}(o)\log n - F_2\{{}_2S_1^{(2)}(\omega)\log p\omega\}$$
$$- nF_2\{{}_2S^{(3)}(\omega)p\omega\log p\omega\}]$$
$$\left. + \frac{a^2}{2}[-{}_2S_1^{(3)}(o)\,2(m+m')\pi\iota + {}_2S_1^{(3)}(o)\log n - F_2\{{}_2S_1^{(3)}(\omega)\log p\omega\}]\right\},$$

the logarithms having their principal values with respect to $-(\omega_1+\omega_2)$.

Now $${}_2S_1'(a) = {}_2S_1'(o) + a\,{}_2S_1^{(2)}(o) + \frac{a^2}{2}\,{}_2S_1^{(3)}(o),$$

for the higher terms in TAYLOR's expansion all vanish. Hence

$$\log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)} = \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz$$
$$+ {}_2S_0(a)\,2(M+m+m')\pi\iota + {}_2S_1'(o)\,2M\pi\iota,$$

which is the expression which has been previously obtained in § 45.

§ 54. We next proceed to consider the values of the double Riemann ζ function, $\zeta_2(s, a\,|\,\omega_1, \omega_2)$, when s is a negative integer.

By the definition of § 39,

$$\zeta_2(s, a\,|\,\omega_1, \omega_2) = \frac{\iota\Gamma(1-s)}{2\pi}\,e^{2Ms\pi\iota}\int_L \frac{e^{-az}(-z)^{s-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

a being positive with respect to the ω's.

Now in § 15 it has been proved that

$$\frac{ze^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2S_1^{(3)}(a)}{z} - {}_2S_1^{(2)}(a) + \frac{{}_2S_1'(a)}{1\,!}z + \ldots$$
$$\ldots + (-)^{n-1}\frac{{}_2S_n'(a)}{n\,!}z^n + \ldots$$

Therefore

$$\zeta_2(s, a\,|\,\omega_1, \omega_2) = -\frac{\iota\Gamma(1-s)}{2\pi}\,e^{2Ms\pi\iota}\int(-z)^{s-2}\,dz\left\{\frac{1}{\omega_1\omega_2 z} - {}_2S_1^{(2)}(a)\right.$$
$$\left. + \frac{{}_2S_1'(a)}{1\,!}z \ldots + (-)^{n-1}\frac{{}_2S_n'(a)}{n\,!}z^n + \ldots\right\},$$

the latter integral being taken along a small circle surrounding the origin. [Since s is an integer, the two line integrals in the standard reduction of § 46 destroy one another.]

Hence if $s = -m$, when m is a positive integer greater than or equal to zero,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \Gamma(1+m)\frac{{}_2S'_{m+1}(a)}{(m+1)!}.$$

Thus, when s is a negative integer,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{{}_2S_{1-s}(a)}{1-s}.$$

But by § 12 corollary, $$\frac{{}_2S'_{n+1}(a)}{n+1} = {}_2B_{n+1}(\omega_1, \omega_2) + {}_2S_n(a),$$

and therefore finally, when s is a negative integer,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = {}_2S_{-s}(a) + {}_2B_{1-s}(\omega_1, \omega_2).$$

When a is not positive with respect to the ω's this formula continues to hold, as is immediately evident by the theory of the function $\zeta_2(s, a \mid \omega_1, \omega_2)$, to which we shall shortly proceed.

§ 55. We proceed next to find the values of $\zeta_2(s, a \mid \omega_1, \omega_2)$ for positive integral (including zero) values of s.

We have seen, in § 48, that when a is positive with respect to the ω's,

$$\frac{\iota}{2\pi}\int(\mathrm{L})\frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S'_1(a),$$

so that $$\zeta_2(o, a \mid \omega_1, \omega_2) = {}_2S'_1(a).$$

Differentiate with regard to a, and we find

$$\frac{\iota}{2\pi}\int(\mathrm{L})\frac{e^{-az}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S'^{(2)}_1(a).$$

Now by § 39, when ϵ is a small quantity,

$$\zeta_2(1-\epsilon, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(\epsilon)}{2\pi}\, e^{-2\mathrm{M}\epsilon\pi\iota}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-\epsilon}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz$$

$$= \frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}\{1-\epsilon\log(-z)+\ldots\}\{1-\gamma\epsilon+\ldots\}\{1-2\mathrm{M}\epsilon\pi\iota\ldots\}}{\epsilon(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

so that, neglecting powers of ϵ above the first,

$$\epsilon\zeta_2(1-\epsilon, a \mid \omega_1, \omega_2) = {}_2S_1^{(2)}(a)\{1-\gamma\epsilon-2\mathrm{M}\epsilon\pi\iota\} - \frac{\iota}{2\pi}\,\epsilon\int_{\mathrm{L}}\frac{\log(-z)e^{-az}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}.$$

But we immediately deduce from § 53 that

$$-\frac{d}{da}\log\Gamma_2(a) = \frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}\log(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \{2\pi\iota(\mathrm{M}+m+m')+\gamma\}\,{}_2S_1^{(2)}(a).$$

Thus $$\underset{s=1}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2S_1^{(2)}(a)}{s-1}\right] = -\psi'_2(a) + 2(m+m')\pi\iota_2 S_1^{(2)}(a).$$

Again differentiating, we have

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S_1^{(3)}(a).$$

Also, if ϵ be small, we have

$$\zeta_2(2-\epsilon, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(\epsilon-1)}{2\pi}\, e^{-2M\epsilon\pi\iota}\int_L \frac{e^{-az}(-z)^{1-\epsilon}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

and $$\Gamma(\epsilon-1) = \frac{\Gamma(\epsilon)}{\epsilon-1} = -\frac{(1-\gamma\epsilon\ \ldots)(1+\epsilon+\ \ldots)}{\epsilon} = -\frac{1}{\epsilon}\{1+(1-\gamma)\epsilon\ \ldots\}.$$

Thus $$\zeta_2(2-\epsilon, a \mid \omega_1, \omega_2) = -\frac{1}{\epsilon\omega_1\omega_2} + \frac{\gamma-1+2M\pi\iota}{\omega_1\omega_2} + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

so that, since

$$\psi_2^{(2)}(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)\log(-z)\,dz}{(1-e^{\omega_1 z})(1-e^{-\omega_2 z})} + [2(M+m+m')\pi\iota+\gamma]_2 S_1^{(3)}(a),$$

we have

$$\underset{s=2}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) - \frac{1}{(s-2)\,\omega_1\omega_2}\right] = \psi_2^{(2)}(a) - \frac{1}{\omega_1\omega_2} - 2(m+m')\pi\iota_2 S_1^{(3)}(a).$$

Tabulating our results, we see that

$$\begin{aligned}
\zeta_2(s, a \mid \omega_1, \omega_2) &= \frac{(-)^s}{(s-1)!}\,\psi_2^{(s)}(a), && \text{when } s > 2\,; \\
&= \frac{{}_2S_1^{(3)}(a)}{s-2} + \psi_2^{(2)}(a) - \frac{1}{\omega_1\omega_2} - 2(m+m')\pi\iota_2 S_1^{(3)}(a), && s = 2\,; \\
&= -\frac{{}_2S_1^{(2)}(a)}{s-1} - \psi_2^{(1)}(a) + 2(m+m')\pi\iota_2 S_1^{(2)}(a), && s = 1\,; \\
&= {}_2S'_1(a) && ,\ s = 0\,; \\
&= {}_2S_{-s}(a) + {}_2B_{1-s}(\omega_1, \omega_2) && ,\ s < 0.
\end{aligned}$$

These formulæ hold for all values of the variable a, though we have only established them for the case when a is positive with respect to the ω's. They evidently agree with the results established for the case $\omega_1 = \omega_2$ in "The Theory of the G Function," § 34.

§ 56. In a note appended to the "Theory of the Gamma Function," it was stated that a theory of the simple Riemann ζ function had been developed by MELLIN.* It

* MELLIN, 'Acta Societatis Fennicae,' vol. 24, No. 10, 1899.

was, however, published after my paper had been sent to press, and I was therefore ignorant of his elegant results. Expressed in the notation which I have adopted, his method is as follows.

It is evident from the expression of the function $\zeta(s, a, \omega)$ as an asymptotic limit, that its importance lies in the fact that it is a solution of the difference equation

$$f(a+\omega)-f(a)=-\frac{1}{a^s},$$

when s has any value, real or complex. [From this result we see at once that we should expect that, when s is a positive integer, the simple ζ function should be substantially a derivative of the gamma function, and when s is a negative integer, a Bernoullian function.] Now when $\mathfrak{R}(s)>1$, the simplest solution of our difference equation is evidently

$$\zeta(s, a, \omega)=\sum_{n=0}^{\infty}\frac{1}{(a+n\omega)^s}.$$

This solution becomes nugatory when $\mathfrak{R}(s) \not> 1$, but MELLIN has succeeded in finding a modified solution by the following ingenious modification of MITTAG-LEFFLER'S process.

We construct the function

$$S_{-s,k}(a \mid \omega)$$

when $1 > \mathfrak{R}(s) < -k$, by writing $-s$ in place of m in the m^{th} simple Bernoullian function

$$S_m(a \mid \omega)=\frac{a^{m+1}}{(m+1)\omega}-\frac{a^m}{2}+\ldots+\binom{m}{r}a^{m-r}{}_1B_{r+1}(\omega)\ldots,$$

and taking the sum of the first $k+1$ terms, k being of course a positive integer.

Thus
$$S_{s,-k}(a \mid \omega)=\frac{a^{1-s}}{(1-s)\omega}-\frac{a^{-s}}{2}+\sum_{r=1}^{k=1}\binom{-s}{r}a^{-s-r}{}_1B_{1+r}(\omega).$$

And now $\zeta(s, a, \omega)$ is defined by the relation

$$\zeta(s, a, \omega)$$
$$=-S_{-s,k}(a \mid \omega)+\sum_{n=0}^{\infty}\left[\frac{1}{(a+n\omega)^s}-S_{-s,k}\{a+(n+1)\omega \mid \omega\}+S_{-s,k}(a+n\omega \mid \omega)\right].$$

We readily see that this function formally satisfies our fundamental difference equation, and we may at once prove that the series does define a function existing over the whole plane.

For when
$$s=0,-1,\ldots,-k,$$

it is evident that
$$S_{-s,k}(a \mid \omega)=S_{-s}(a \mid \omega)+\text{constant},$$

and therefore
$$S_{-s,k}(a \mid \omega)-S_{-s,k}(a)-a^{-s}=0.$$

When a is large and s has any value, the left-hand side of this relation may be expanded in the form

$$\frac{P_0(s)}{a^s} + \frac{P_1(s)}{a^{s+1}} + \ldots + \frac{P_r(s)}{a^{r+s}} + \ldots,$$

where the P's are integral polynomials of s of degree indicated by their suffices. As they vanish for $(k+1)$ values of s, we must have

$$S_{-s,k}(a+\omega) - S_{-s,k}(a) - a^{-s} = \frac{P_{k+1}(s)}{a^{s+k+1}} + \frac{P_{k+2}(s)}{a^{s+k+2}} + \ldots$$

Thus $\zeta(s, a, \omega)$ is convergent with $\sum_{n=1}^{\infty} \frac{1}{(a+n\omega)^{s+k+1}}$, and is therefore convergent, provided

$$\mathfrak{R}(s+k+1) > 1, \qquad \text{or } \mathfrak{R}(s) > -k,$$

which is the condition with which we started.

There is one point which does not arise in the work of MELLIN, who takes the case $\omega = 1$. It is that throughout we must work with many-valued functions with s as index, which have their principal values with respect to the axis of $-\omega$. For in expanding $S_{-s,k}(a+\omega) - S_{-s,k}(a) - a^{-s}$, where a is large, we have tacitly assumed that $\log(a+\omega) = \log a + \log\left(1 + \frac{\omega}{a}\right)$, which, unless ω is real, is not the case when a is large and nearly real and negative, so that a and $a + \omega$ lie on opposite sides of the axis of -1, if this axis is the axis of the logarithms.

§ 57. It is now possible to construct the double Riemann ζ function by extending the previous analysis. The function so constructed might be made fundamental in the theory of double gamma functions and double Bernoullian numbers, these functions arising for particular values of the variable s. We will indicate the development of the theory from this point of view, for brevity establishing only the principal results, or those which, as in §§ 54, 55, have been established only over part of the a plane.

The double Riemann ζ function $\zeta_2(s, a \mid \omega_1, \omega_2)$ is the simplest solution of the difference equation

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = \frac{1}{a^s}$$

a, s, ω_1 and ω_2 having any complex values such that ω_2/ω_1 is not real and negative. The determination of a^{-s} will appear in the course of the investigation.

In the first place, it is evident that when $\mathfrak{R}(s) > 2$, a solution is given by the series

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s},$$

which will then, by EISENSTEIN's theorem, be convergent.

When $2 > \mathfrak{R}(s) > -k$, k being some positive integer, we form a modified solution by the introduction of the function ${}_2S_{-s,k}(a \mid \omega_1, \omega_2)$, formed as follows. We take the mth double Bernoullian function, ${}_2S_m(a \mid \omega_1, \omega_2)$, write $-s$ in place of m, and then take the sum of the first $(k+2)$ terms.

We thus have

$$ {}_2S_{-s,k}(a \mid \omega_1, \omega_2) = \frac{1}{(1-s)(2-s)\,\omega_1\omega_2\,a^{s-2}} - \frac{\omega_1+\omega_2}{2(1-s)\,\omega_1\omega_2\,a^{s-1}} + \frac{{}_2B_1}{a^s} + \sum_{r=0}^{k-1}\binom{-s}{r}\frac{{}_2B_{r+1}}{a^{s+r}}, $$

and then $\zeta_2(s, a \mid \omega_1, \omega_2)$ is given by

$$ \begin{aligned} {}_2S_{-s,k}(a) - \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \big[& {}_2S_{-s,k}\{a+(m_1+1)\omega_1+(m_2+1)\omega_2\} \\ & - {}_2S_{-s,k}\{a+(m_1+1)\omega_1+m_2\omega_2\} \\ & - {}_2S_{-s,k}\{a+m_1\omega_1+(m_2+1)\omega_2\} + {}_2S_{-s,k}\{a+m_1\omega_1+m_2\omega_2\} \\ & -(a+m_1\omega_1+m_2\omega_2)^{-s}\big], \end{aligned} $$

an expression which for shortness we shall sometimes write

$$ {}_2S_{-s,k}(a) - \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \chi(a+m_1\omega_1+m_2\omega_2 \mid s, k). $$

It is at once evident that the function so defined formally satisfies the fundamental difference equation, and we may readily prove that the series is in general convergent.

For when $s = 0, -1, -2, \ldots, -k, -(k+1)$, obviously

$$ {}_2S_{-s,k}(a \mid \omega_1, \omega_2) = {}_2S_{-s}(a \mid \omega_1, \omega_2) + \lambda a + \mu $$

where λ and μ are constants.

And therefore

$$ \chi(z \mid s,k) \text{ vanishes when } s = 0, -1, -2, \ \ldots, -k, -(k+1). $$

When z is large this expression admits of expansion in the form

$$ \frac{P_0(s)}{z^s} + \frac{P_1(s)}{z^{s+1}} + \ldots $$

where $P_0(s)$, $P_1(s)$, ... are integral polynomials in s of degree indicated by their suffixes, provided that the logarithms which intervene in defining the many-valued functions with s as index are such that, when ϵ is small compared with z,

$$ \log z + \log\left(1+\frac{\epsilon}{z}\right) = \log(z+\epsilon). $$

If the axes of ω_1 and ω_2 include the axis of -1, this will not be the case for terms of the double series, for which the numbers m_1 and m_2 in the term $z = a + m_1\omega_1 + m_2\omega_2$ are large, unless the logarithms have their principal value with respect to some line between the axes of $-\omega_1$ and $-\omega_2$. We take this line to be $-(\omega_1+\omega_2)$.

And now, since $\chi(z|s, k)$ vanishes for $k+2$ values of s, we see that its expansion when z is large must be

$$\frac{P_{k+2}(s)}{z^{s+k+2}} + \frac{P_{k+3}(s)}{z^{s+k+3}} + \dots.$$

The series for $\zeta_2(s, a|\omega_1, \omega_2)$ is therefore convergent with

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^{s+k+2}}.$$

It is then convergent when $s + k + z > 2$ or $\mathfrak{R}(s) > -k$, $|s|$ being finite.

We have then obtained a solution of the difference equation

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f(a + \omega_2) + f(a) = a^{-s},$$

where a^{-s} has its principal value with respect to the axis of $-(\omega_1 + \omega_2)$, which is valid for all values of s, a, ω_1 and ω_2.

§ 58. The identity of the function $\zeta_2(s, a|\omega_1, \omega_2)$ just defined with that previously employed is easily seen.

From the mode of formation of ${}_2S_{-s,k}(a|\omega_1, \omega_2)$ it is evident that when a is positive with respect to the ω's, we have

${}_2S_{-s,k}(a|\omega_1, \omega_2) =$ the sum of the first $(k+2)$ terms in the expansion in powers of $\frac{1}{a}$ of

$$\frac{\iota\Gamma(1-s)}{2\pi} e^{2M\pi\iota s} \int_L \frac{e^{-az}(-z)^{s-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} dz,$$

and therefore

$${}_2S_{-s,\infty}(a|\omega_1, \omega_2) = \frac{\iota\Gamma(1-s)}{2\pi} e^{2M\pi\iota s} \int_L \frac{e^{-az}(-z)^{s-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} dz.$$

Therefore when n is large we have the asymptotic expansion

$${}_2S_{-s,\infty}\{a + (pn+1)\omega_1|\omega_1, \omega_2\} = \frac{1}{(s-1)(s-2)\omega_1\omega_2(pn\omega_1)^{s-2}} - \frac{1}{s-1} \cdot \frac{2a + \omega_1 + \omega_2}{2\omega_1\omega_2(pn\omega_1)^{s-1}}$$
$$+ \frac{1}{s-1 \,.\, \omega_1} \cdot \frac{1}{(pn\omega_1)^{s-1}} + \sum_{m=1}^{\infty} (-)^{m-1} \frac{s(s+1)\dots(s+m-2)}{m!} \frac{{}_2S'_m(a+\omega_1)}{(pn\omega_1)^{m+s-1}},$$

a formula which may be proved to be true for all values of a by a term-by-term expansion of the series for ${}_2S_{-s,\infty}\{a + (pn+1)\omega_1|\omega_1, \omega_2\}$.

Now from the expression for $\zeta_2(s, a|\omega_1, \omega_2)$ given in § 57, we see on taking the $(pn+1)$ first values of m_1 and the $(qn+1)$ first values of m_2, that

$$\zeta_2(s, a|\omega_1\omega_2) = \underset{n=\infty}{Lt} \left[\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s} - {}_2S_{-s,k}\{a + (pn+1)\omega_1 + (qn+1)\omega_2\} \right.$$
$$\left. + {}_2S_{-s,k}\{a + (pn+1)\omega_1\} + {}_2S_{-s,k}\{a + (qn+1)\omega_2\} \right].$$

Putting $k = \infty$, and employing the asymptotic expansion just obtained for ${}_2S_{-s,\infty}\{a + (pn + 1\}\,\omega_1]$ we obtain $\zeta_2(s, a \mid \omega_1, \omega_2)$ from the same asymptotic equality as that by which it has previously been defined in §§ 39 and 41.

§ 59. As an example of the way in which we should proceed in a theory based on the double gamma function as defined in § 57, we will prove the relation

$$\underset{s=1}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a)\right] = -\,\psi_2'(a \mid \omega_1, \omega_2) + 2(m + m')\pi\iota\,{}_2S_0\,(a)$$

established in § 55 for the case when a is positive with respect to the ω's.

In the first place, when $s = 1 + \epsilon$ and ϵ is small, we see that

$$\begin{aligned}{}_2S_{-s,k}(a \mid \omega_1, \omega_3) &= {}_2S_{-1-\epsilon,0}(a \mid \omega_1, \omega_2)\\ &= \frac{1}{-\,\epsilon(1-\epsilon)\omega_1\omega_2 a^{\epsilon-1}} + \frac{\omega_1 + \omega_2}{2\epsilon\omega_1\omega_2}\,a^{-}\\ &= \left(-\frac{1}{\epsilon} + \log a\right){}_2S_0'(a) - \frac{a}{\omega_1\omega_2} + \text{higher powers of } \epsilon.\end{aligned}$$

Therefore taking $\zeta_2(s, a \mid \omega_1, \omega_2)$ as the limit, when n is infinite, of the sum obtained by taking the first $(pn + 1,\ qn + 1)$ terms of the double series

$$\begin{aligned}&\zeta_2(1 + \epsilon, a) + \frac{1}{\epsilon}\,{}_2S_0'(a)\\ &= \underset{n=\infty}{\mathrm{L}t}\Bigg[\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{a + m_1\omega_1 + m_2\omega_2} - \frac{a}{\omega_1\omega_2}\\ &\qquad - {}_2S_0'[a + (pn+1)\omega_1 + (qn+1)\omega_2]\log[a + (pn+1)\omega_1 + (qn+1)\omega_2]\\ &\qquad + {}_2S_0'[a + (pn+1)\omega_1]\log[a + (pn+1)\omega_1]\\ &\qquad + {}_2S_0'[a + (qn+1)\omega_2]\log[a + (qn+1)\omega_2]\Bigg],\end{aligned}$$

the logarithms having their principal values with respect to the axis of $-(\omega_1 + \omega_2)$.

Putting $a = 0$, $p = q = 1$, we find

$$\begin{aligned}&\underset{\substack{a=0\\ s=1}}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a) - \frac{1}{a}\right]\\ &= \sum_0^n{\sum}'\frac{1}{m_1\omega_1 + m_2\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - (n+1)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2)\\ &\qquad + \frac{n+1}{\omega_2}\log\omega_1 + \frac{n+1}{\omega_1}\log\omega_2 + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\\ &= \gamma_{22}(\omega_1, \omega_2) - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\,2\pi\iota(m + m').\end{aligned}$$

And now, in the limit where n is infinite, we find

$$\underset{s=1}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a)\right]$$

$$= \sum_0^n \sum \frac{1}{a + m_1\omega_1 + m_2\omega_2} - \left\{\frac{a + (n+1)\omega_1 + (n+1)\omega_2}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right\}\log n(\omega_1+\omega_2)$$

$$+ \left\{\frac{a + (n+1)\omega_2}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right\}\log n\omega_2 + \left\{\frac{a + (n+1)\omega_1}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right\}\log \omega_1$$

$$= \frac{1}{a} + \sum_0^n \sum' \left[\frac{1}{a+\Omega} - \frac{1}{\Omega} + \frac{a}{\Omega^2}\right] + a\gamma_{21} + \gamma_{22} + 2(m+m')\pi\iota \, . \, \frac{2a - \omega_1 - \omega_2}{2\omega_1\omega_2}$$

where $\Omega = m_1\omega_1 + m_2\omega_2$,

$$= -\psi_2^{(1)}(a \mid \omega_1, \omega_2) + {}_2S_0'(a)\,2(m+m')\pi\iota,$$

the complete form of the result established in § 55 for the case where a is positive with respect to the ω's.

We may establish the other results of that paragraph in a similar manner.

§ 60. Finally we will briefly consider the reduction of the second form of the double ζ function to the double gamma function in the case when $s = 0$.

If we put $s = \epsilon$, where $|\epsilon|$ is very small, we obtain

$${}_2S_{-s,k}(a \mid \omega_1, \omega_2) = {}_2S_{-\epsilon,1}(a \mid \omega_1, \omega_2)$$

$$= \frac{a^2}{2\omega_1\omega_2}\{1 + \epsilon + \ldots\}\left\{1 + \frac{\epsilon}{2} + \ldots\right\}\{1 - \epsilon \log a + \ldots\}$$

$$- \frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2}\{1 + \epsilon + \ldots\}\{1 - \epsilon\log a + \ldots\}$$

$$+ {}_2B_1(\omega_1, \omega_2)\{1 - \epsilon \log a + \ldots\}$$

$$= {}_2S_1'(a)[1 - \epsilon\log a] + \epsilon\left[\frac{3a^2}{4\omega_1\omega_2} - a\frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right] + \text{higher powers of } \epsilon.$$

And therefore, by the second definition of the double ζ function

$$\zeta_2(\epsilon, a \mid \omega_1, \omega_2) = {}_2S_1'(a)[1 - \epsilon \log a] + \epsilon\left[\frac{3a^2}{4\omega_1\omega_2} - a\frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right]$$

$$- \sum_{m_1=0}\sum_{m_2=0}^{\infty}\Big[{}_2S_1'(a+\Omega+\omega_1+\omega_2)\{1 - \epsilon\log(a+\Omega+\omega_1+\omega_2)\ldots\}$$

$$- {}_2S_1'(a+\Omega+\omega_1)\{1 - \epsilon\log(a+\Omega+\omega_1)\ldots\}$$

$$- {}_2S_1'(a+\Omega+\omega_2)\{1 - \epsilon\log(a+\Omega+\omega_2)\ldots\}$$

$$+ {}_2S_1'(a+\Omega)\{1 - \epsilon\log(a+\Omega)\ldots\} - 1 + \epsilon\log(a+\Omega) + \frac{3\epsilon}{2}\Big],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$.

Thus $$\zeta_2(o, a \mid \omega_1, \omega_2) = {}_2S_1'(a)$$

and

$$\underset{\epsilon=o}{\mathrm{Lt}}\left[\frac{\zeta_2(\epsilon, a \mid \omega_1, \omega_2) - {}_2S_1'(a)}{\epsilon}\right] = -\sum_{m_1=0}^{m}\sum_{m_2=0}^{m}\log(a + m_1\omega_1 + m_2\omega_2) +$$

[OVER]

$$+\,{}_2S_1'[a+(pn+1)\omega_1+(qn+1)\omega_2]\log[a+(pn+1)\omega_1+(qn+1)\omega_2]$$
$$-\,{}_2S_1'[a+(pn+1)\omega_1]\log[a+(pn+1)\omega_1]$$
$$-\,{}_2S_1'[a+(qn+1)\omega_2]\log[a+(qn+1)\omega_2]$$
$$+\frac{3a^2}{4\omega_1\omega_2}-\frac{a}{2}\cdot\frac{\omega_1+\omega_2}{\omega_1\omega_2}-\frac{3}{2}(pn+1)(qn+1)$$

in the limit when n is made infinite.

Substituting the value of ${}_2S_1'(a)$ in the various terms, expanding the logarithms, and re-arranging the result in powers of n, we find, with the symbolic notation of § 49,

$$\underset{s=0}{\mathrm{L}t}\left[\frac{\zeta_2(s,a\,|\,\omega_1,\omega_2)-{}_2S_1'(a)}{s}\right]$$
$$=-\log\prod_{\substack{0\\m_1}}^{pn}\prod_{\substack{0\\m_2}}^{qn}(a+\Omega)+pq[n^2\log n-n^2(\tfrac{1}{1}+\tfrac{1}{2})]+(p+q)[n\log n-n]$$
$$+\frac{n^2}{2}F_2[{}_2S_1^{(3)}(\omega)(p\omega)^2\log p\omega]+nF_2[{}_2S_1^{(2)}(a+\omega)p\omega\log p\omega]$$
$$+F_2[{}_2S_1'(a+\omega)\log p\omega]+[1-{}_2S_1(a)]\log n$$

in the limit when n is made infinite.

Since the left-hand side of this equality is by the definition of $\zeta_2(s,a\,|\,\omega_1,\omega_2)$ finite unless a is at one of the points

$$-(m_1\omega_1+m_2\omega_2)\qquad \left.\begin{matrix}m_1=0,1,2,\ldots,\infty\\ m_2=0,1,2,\ldots,\infty\end{matrix}\right\},$$

we see that we have thus been led in a purely algebraical manner to the determination of the dominant terms of the fundamental expansion of § 49.

If we make $a=0$, and remember the definition of § 50, viz. :—

$$-\log\rho_2(\omega_1,\omega_2)=\underset{\substack{s=0\\a=0}}{\mathrm{L}t}\left[\frac{\zeta_2(s,a\,|\,\omega_1,\omega_2)-{}_2S_1'(a)}{s}+\log a\right],$$

we arrive at the dominant terms of the extension of STIRLING'S theorem to two parameters.

If we utilise this result in conjunction with the one just obtained, we find

$$\underset{s=0}{\mathrm{L}t}\left[\frac{\zeta_2(s,a)-{}_2S_1'(a)}{s}\right]-\log\rho_2(\omega_1,\omega_2)$$
$$=-\log a+\log\prod_0^{pn}\prod_0^{qn}{}'\,\Omega-\log\prod_0^{pn}\prod_0^{qn}{}'(a+\Omega)+\frac{na}{\omega_1\omega_2}F_2[p\omega\log p\omega]$$
$$+F_2[\{{}_2S_0(a+\omega)-{}_2S_0(\omega)\}\log p\omega]-{}_2S_0(a)\log n,$$

and by the definition of the double gamma function the expression last written reduces to

$$\log \Gamma_2(a) - \left[\frac{a^2}{\omega_1\omega_2} - a\frac{\omega_1+\omega_2}{2\omega_1\omega_2}\right] 2(m+m')\pi\iota$$

Thus
$$\exp.\left\{\operatorname*{Lt}_{s=0} \frac{\zeta_2(s,a) - {}_2S_1'(a)}{s}\right\} = \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} e^{-{}_2S_0(a)\, 2(m+m')\pi\iota}.$$

We may utilise this formula to establish the fundamental difference equations for the double gamma function.

By the definition of the double ζ function of § 57

$$\zeta_2(a+\omega_1) - \zeta_2(a) = \operatorname*{Lt}_{n=\infty}\left[{}_2S_{-s,k}[a+\omega_1+(n+1]\,\omega_2) - {}_2S_{-s,k}[a+(n+1)\,\omega_2] - \sum_{m_2=0}^{n} \frac{1}{(a+m_2\omega_2)^s}\right]$$

Therefore, in the limit when n is infinite,

$$\operatorname*{Lt}_{s=0}\left[\frac{\zeta_2(a+\omega_1) - {}_2S_1'(a+\omega_1)}{s} - \frac{\zeta_2(a) - {}_2S_1'(a)}{s}\right]$$
$$= \sum_{m_2=0}^{n} \log(a+m_2\omega_2) - {}_2S_1'[a+\omega_1+(n+1)\,\omega_2]\log[a+\omega_1+(n+1)\,\omega_2]$$
$$+ {}_2S_1'[a+(n+1)\,\omega_2]\log[a+(n+1)\,\omega_2] + \frac{3\omega_1}{4\omega_2} + \frac{3}{2\omega_2}[a+(n+1)\,\omega_2] - \frac{\omega_1+\omega_2}{2\omega_2}.$$

On reduction we obtain

$$\log\left\{\frac{\Gamma_2(a+\omega_1)}{\Gamma_2(a)} e^{-2(m+m')\pi\iota\, S_1'(a \mid \omega_2)}\right\}$$
$$= \operatorname*{Lt}_{n=\infty}\left[\sum_{m_2=0}^{n} \log(a+m_2\omega_2) - \left(\frac{a+(n+1)\,\omega_2}{\omega_2} - \frac{1}{2}\right)\log n\omega_2 + n\right].$$

This latter expression is, by the expansion obtained in the "Theory of the Gamma Function," § 30, equal to

$$-\log\frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} - 2m'\pi\iota\, S_1'(a \mid \omega_2).$$

[The term $2m'\pi\iota(n+1)$ which arises is absorbed by the identities which change $\log(a+m_2\omega_2)$ into $\log m_2\omega_2 + \log\left(1+\frac{a}{m_2\omega_2}\right)$. The prescription of the absolute logarithms has been throughout left indeterminate.]

We thus have

$$\frac{\Gamma_2^{-1}(a+\omega_1)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota\, S_1'(a \mid \omega_2)},$$

one of the fundamental formulæ of § 23.

Sufficient indication has perhaps now been given of the alternative development of the theory of the double ζ function.

Note.—The asymptotic expansions of this part were obtained in my original cast of this theory, to which reference has already been made in the note which follows the "Theory of the Gamma Function," by the assumption that they would involve merely powers of n and $\log n$ coupled with inductive processes. Such a method, though long, is, could the fundamental assumption be justified, probably the most elementary way of obtaining these results.

[*Additional Note added July* 5, 1900.—Dr. HOBSON has pointed out that, if we admit the validity of the application of the calculus of operators to a parameter in the subject of integration of a contour integral, the theory from § 57 onwards may be developed in a very elegant manner.

We take the formula

$$f(a+\omega_1+\omega_2)-f(a+\omega_1)-fa+\omega_2)+f(a)=\frac{1}{a^s},$$

and the known theorem

$$\frac{1}{a^s}=\frac{\iota\Gamma(1-s)}{2\pi}\int e^{-az}(-z)^{s-1}dz,$$

and deduce

$$f(a)=\frac{\iota\Gamma(1-s)}{2\pi}\frac{1}{(e^{\omega_1\frac{d}{da}}-1)(e^{\omega_2\frac{d}{da}}-1)}\int e^{-az}(-z)^{s-1}dz$$

$$=\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-az}(-z)^{s-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz.]$$

PART IV.

The Multiplication, Transformation, and Integral Formulæ for the Double Gamma Function.

§ 61. After the developments of Part III., we now return to the pure theory of the double gamma function. As regards the multiplication and transformation theories, two distinct courses are open to us. We may either proceed entirely algebraically, making use of the limit theorems which have been established, and so deduce the required results without the intervention of contour integrals at any stage, *or* we may directly utilise these latter to obtain the formulæ in question. The former course is, on abstract grounds, preferable :* we ought to deduce algebraical results by algebraical processes. But it is open to the fatal objection of leading to very lengthy algebra. We will employ the two methods, side by side, to deduce the multiplication formulæ, and it will be observed that the second method is both more elegant and more speedy. For the sake of brevity, the results of the transformation

* In the first sketch of this theory, before the discovery of the contour integral expressions, all the results were obtained in this way.

theory are obtained solely by this course. Inasmuch as the function $\Gamma_2(mz \mid \omega_1, \omega_2)$ can be expressed in terms of the function $\Gamma_2\left(z \,\middle|\, \frac{\omega_1}{m}, \frac{\omega_2}{m}\right)$, the multiplication theory can be deduced from the theory of transformation. As the work of obtaining the new expression is in every case almost equal to that of obtaining the results *ab initio*, we adopt the latter course.

Multiplication Theory.

§ 62. We have from the definition (§ 19)

$$\psi_2^{(3)}(mz) = -2\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{(mz+m_1\omega_1+m_2\omega_2)^3},$$

and therefore

$$m^3\psi_2^{(3)}(mz) = -2\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{\left(z+\frac{m_1\omega_1}{m}+\frac{m_2\omega_2}{m}\right)^3}$$

$$= -2\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{\left(z+\frac{r\omega_1}{m}+\frac{s\omega_2}{m}+m_1\omega_1+m_2\omega_2\right)^3}$$

$$= \sum_{r=0}^{m-1}\sum_{s=0}^{m-1}\psi_2^{(3)}\left(z+\frac{r\omega_1+s\omega_2}{m}\right),$$

the parameters being understood to be ω_1 and ω_2 when not explicitly written.

Integrate with respect to z and we obtain

$$m^2\psi_2^{(2)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(2)}\left(z+\frac{p\omega_1+q\omega_2}{m}\right)+r \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

where r is constant with respect to z.

Now
$$\psi_2^{(2)}(z) = -\gamma_{21}(\omega_1, \omega_2) + \frac{1}{z^2} + \sum_0^{\infty}\sum{}'\left\{\frac{1}{(z+\Omega)^2}-\frac{1}{\Omega^2}\right\}.$$

Substitute from this relation in the identity (1), take the same number of terms involving z on both sides of the equality, and remember that $\frac{1}{(z+\Omega)^2}-\frac{1}{\Omega^2}$ is always to be regarded as a single entity. We find that, in the limit when n is infinite, we have

$$-r + m^2\sum_{\substack{0\\m_1}}^{n}\sum_{\substack{0\\m_2}}^{n}{}'\frac{1}{\Omega^2} = m^2\sum_{\substack{0\\m_1}}^{mn+m-1}\sum_{\substack{0\\m_2}}^{mn+m-1}{}'\frac{1}{\Omega^2}$$

where $\Omega = m_1\omega_1 + m_2\omega_2$.

Now we have seen that (§ 22), in the limit when n is infinite,

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} = -\gamma_{21}(\omega_1, \omega_2) + \frac{1}{\omega_1\omega_2}\log n + \frac{1}{\omega_1\omega_2}[\log\omega_1 + \log\omega_2 - \log(\omega_1+\omega_2)]$$

the principal values of the logarithms being taken.

Therefore
$$r = -\frac{m^2}{\omega_1\omega_2}\log m,$$
so that we have
$$m^2\psi_2^{(2)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(2)}\left(z + \frac{p\omega_1 + q\omega_2}{m}\right) - \frac{m^2}{\omega_1\omega_2}\log m.$$

Integrate again with respect to z and we find

$$m\psi_2^{(1)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(1)}\left(z + \frac{p\omega_1 + q\omega_2}{m}\right) - \frac{m^2 z}{\omega_1\omega_2}\log m + s \quad . \quad . \quad . \quad \text{(ii.)},$$

where s is constant with respect to z.

Now we have, in the limit when n is infinite,

$$\psi_2^{(1)}(z) = -z\gamma_{21}(\omega_1, \omega_2) - \gamma_{22}(\omega_1, \omega_2) - \frac{1}{z} - \sum_{\substack{0\\m_1}}^{n}\sum_{\substack{0\\m_2}}^{n}{}'\left[\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right].$$

Hence we find, on equating the irresoluble terms involving z in the same way on both sides of the identity (ii.), that, in the limit when n is infinite,

$$m\left\{\sum_{\substack{0\\m_1\ m_2}}^{mn+m-1}\sum{}' \frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\} = s + m^2\left\{\sum_{\substack{0\\m_1\ m_2}}^{n}\sum{}' - \gamma_{22}(\omega_1, \omega_2)\right\}$$
$$- \left[\gamma_{21} + \sum_{\substack{0\\m_1\ m_2}}^{n}\sum{}'\frac{1}{\Omega^2}\right]\sum_{\substack{0\\p\ \ q}}^{m-1}\sum{}'\left(\frac{p\omega_1 + q\omega_2}{m}\right),$$

and, therefore, when n is infinite,

$$s = m\left\{\sum_{0}^{mn+m-1}\sum{}'\frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\} - m^2\left\{\sum_{0}^{n}\sum{}'\frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\}$$
$$+ m(m-1)\frac{\omega_1 + \omega_2}{2}\left\{\sum_{0}^{n}\sum{}'\frac{1}{\Omega^2} + \gamma_{21}(\omega_1, \omega_2)\right\}.$$

Now in § 23 we have seen that

$$\sum_{\substack{0\\m_1\ m_2}}^{n}\sum{}'\frac{1}{\Omega} - \gamma_{22} = \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n + (n+1)\left[\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2) - \frac{1}{\omega_2}\log\omega_1\right.$$
$$\left. - \frac{1}{\omega_1}\log\omega_2 - \frac{2m\pi\iota}{\omega_2} - \frac{2m'\pi\iota}{\omega_1}\right] - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\},$$

and, therefore, after some reduction, we see that

$$s = m\frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log m.$$

We thus have

$$m\,\psi_2^{(1)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(1)}\left(z+\frac{p\omega_1+q\omega_2}{m}\right) - \frac{m^2 z}{\omega_1\omega_2}\log m + m\,\frac{(\omega_1+\omega_2)}{2\omega_1\omega_2}\log m.$$

Integrate again with respect to z, determine the constant by making $z=0$, and remember that $\underset{z=0}{Lt}\,\frac{\Gamma_2(mz)}{\Gamma_2(z)} = \frac{1}{m}$: we obtain the formula

$$m\,\Gamma_2(mz) = \frac{\prod\limits_{r=0}^{m-1}\prod\limits_{s=0}^{m-1}\Gamma_2\left(z+\frac{r\omega_1+s\omega_2}{m}\right)}{\prod\limits_{r=0}^{m-1}\prod\limits_{s=0}^{m-1}{}'\,\Gamma_2\left(\frac{r\omega_1+s\omega_2}{m}\right)}\,e^{-{}_2S_0(mz)\log m},$$

the principal value of the logarithm being taken.

§ 63. By means of the extension of STIRLING'S theorem to two parameters (§ 50), it is possible to express the form $\prod\limits_{r=0}^{m-1}\prod\limits_{s=0}^{m-1}{}'\,\Gamma_2\left(\frac{r\omega_1+s\omega_2}{m}\right)$ which has thus arisen in terms of the double Stirling function $\rho_2(\omega_1,\omega_2)$.

For, in the limit when n is infinite,

$$\Gamma_2^{-1}(z) = e^{\frac{z^2}{2}\gamma_{21}+z\gamma_{22}}\,.\,z\,.\prod_{\substack{0\\m_1}}^{n}\prod_{\substack{0\\m_2}}^{n}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right],$$

and therefore, under the same condition, we obtain from the result of the previous paragraph

$$mz\,e^{\gamma_{21}\frac{m^2z^2}{2}+mz\gamma_{22}}\prod_{\substack{0\\m_1}}^{mn+m-1}\prod_{m_2}{}'\left[\left(1+\frac{mz}{\Omega}\right)e^{-\frac{mz}{\Omega}+\frac{1}{2}\frac{m^2z^2}{\Omega^2}}\right]$$

$$=\frac{m e^{{}_2S_0(mz)\log m}}{\prod\limits_{r=0}^{m-1}\prod\limits_{s=0}^{m-1}{}'\,\Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right)}\cdot\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left[\left(z+\frac{r\omega_1+s\omega_2}{m}\right)e^{\gamma_{21}\frac{\left(z+\frac{r\omega_1+s\omega_2}{m}\right)^2}{2}+\gamma_{22}\left(z+\frac{r\omega_1+s\omega_2}{m}\right)}\right.$$

$$\left.\times\prod_{\substack{0\\m_1}}^{n}\prod_{\substack{0\\m_2}}^{n}\left\{\left(1+\frac{z+\frac{r\omega_1+s\omega_2}{m}}{\Omega}\right)e^{-\frac{z+\frac{r\omega_1+s\omega_2}{m}}{\Omega}+\frac{1}{2}\frac{\left(z+\frac{r\omega_1+s\omega_2}{m}\right)^2}{\Omega^2}}\right\}\right]$$

Make now $z=0$, and we find

$$\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}'\,\Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right) = \prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\left(\frac{r\omega_1+s\omega_2}{m}\right)^1\prod_{\substack{0\\m_1}}^{n}\prod_{\substack{0\\m_2}}^{n}{}'\left(1+\frac{\frac{r\omega_1+s\omega_2}{m}}{\Omega}\right)\right.$$

$$\left.\times\,\text{Exp.}\left[-\frac{r\omega_1+s\omega_2}{m}\left(\sum_{\substack{0\\m_1}}^{n}\sum_{m_2}{}'\frac{1}{\Omega}-\gamma_{22}\right)+\frac{1}{2}\left(\frac{r\omega_1+s\omega_2}{m}\right)^2\left(\sum_{\substack{0\\m_1}}^{n}\sum_{m_2}{}'\frac{1}{\Omega^2}+\gamma_{21}\right)\right]\right\}$$

where $\left(\frac{r\omega_1+s\omega_2}{m}\right)^1$ denotes that in the product the term for which $\left.\begin{matrix}r=0\\s=0\end{matrix}\right\}$ simultaneously is to be excluded.

We have then, in the limit when n is infinite,

$$\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}' \Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right) = \frac{\prod_0^{mn+m-1}\prod' \frac{1}{\Omega}}{\left[\prod_0^{n}\prod' \frac{1}{\Omega}\right]^{m^2}} \cdot m^{-m^2n^2-2m^2n-m^2+1}.$$

$$\times \text{Exp.}\left[-m(m-1)\frac{\omega_1+\omega_2}{2}\left(\Sigma_0^n\Sigma'\frac{1}{\Omega}-\gamma_{22}\right)+\left\{\frac{(m-1)(2m-1)}{12}(\omega_1^2+\omega_2^2)\right.\right.$$
$$\left.\left.+\frac{(m-1)^2}{4}\omega_1\omega_2\right\}\left\{\Sigma_0^n\Sigma'\frac{1}{\Omega^2}+\gamma_{21}\right\}\right].$$

Utilise now the extension of STIRLING'S theorem and the limit formulæ for $\Sigma_0^n\Sigma'\frac{1}{\Omega}$ and $\Sigma_0^n\Sigma'\frac{1}{\Omega^2}$. We find that

$$\log\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}'\Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right)=\{-m^2n^2-2m^2n-m^2+1\}\log m$$
$$+(mn+m-1)^2\log(mn+m-1)-\tfrac{3}{2}(mn+m-1)^2$$
$$+(mn+m-1)^2F_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]+2(mn+m-1)\log(mn+m-1)$$
$$-2(mn+m-1)+(mn+m-1)\,F_2[{}_2S_1^{(2)}(\omega)\,\omega\log\omega]$$
$$+[1-{}_2S_1'(o)]\log(mn+m-1)+(1-m^2)\log\rho_2(\omega_1,\omega_2)+F_2[{}_2S_1'(\omega)\log\omega]$$
$$-m^2\left[n^2\log n-\frac{3n^2}{2}\right]-m^2n^2F_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-2m^2[n\log n-n]$$
$$-m^2n\,\text{Fe}[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-m^2[1-{}_2S_1'(o)]\log n-m^2F_2[{}_2S_1'(\omega)\log\omega]$$
$$-m(m-1)\frac{\omega_1+\omega_2}{2}\left\{{}_2S_1^{(2)}(o)\,2(m+m')\pi\iota+nF_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-{}_2S_1^{(2)}(o)\log n\right.$$
$$\left.+F_2[{}_2S_1^{(2)}(\omega)\log\omega]\right\}+\left[\frac{2m^2-3m+1}{12}(\omega_1^2+\omega_2^2)+\frac{m^2-2m+1}{4}\omega_1\omega_2\right]$$
$$\times\left\{-2(m+m')\pi\iota\,{}_2S_1^{(3)}(o)+\frac{1}{\omega_1\omega_2}\log n-F_2[{}_2S_1^{(3)}(\omega)\log\omega]\right\},$$

where the logarithms have their principal values with respect to the axis of $-(\omega_1+\omega_2)$.

The labour of reducing such an expression is evidently very great. It is diminished by observing that the result must be independent of n, so that we may neglect all terms which involve this letter; but even then it is only after several steps that we prove that the right-hand side is equal to

$$[1-{}_2S_1'(o)]\log m+(1-m^2)\log\rho_2(\omega_1,\omega_2)$$
$$-2(m+m')\pi\iota\left[-(m^2-m)\frac{(\omega_1+\omega_2)^2}{4\omega_1\omega_2}+\frac{2m^2-3m+1}{12}\cdot\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}+\frac{m^2-2m+1}{4}\right]$$
$$=[1-{}_2S_1'(o)]\log m+(1-m^2)\log\rho_2(\omega_1,\omega_2)+(m^2-1)\,2(m+m')\pi\iota\,{}_2S_1(o)$$

We thus see that

$$\prod_{r=0}^{m-1} \prod_{s=0}^{m-1}{}' \Gamma_2\left(\frac{r\omega_1 + s\omega_2}{m}\right) = [\rho_2(\omega_1, \omega_2)]^{m^2-1} \cdot m^{{}_2S_1'(o)-1}, e^{(1-m^2)2(m+m')\pi\iota_2 S_1'(o)}.$$

§ 64. In the case when $m = 2$ the preceding result has an especial importance. It is convenient to write

$$\Gamma_2\left(z + \frac{\omega_1}{2}\,\middle|\,\omega_1, \omega_2\right) = \gamma_1(z \mid \omega_1, \omega_2),$$

$$\Gamma_2\left(z + \frac{\omega_2}{2}\,\middle|\,\omega_1, \omega_2\right) = \gamma_2(z \mid \omega_1, \omega_2),$$

$$\Gamma_2\left(z + \frac{\omega_1 + \omega_2}{2}\,\middle|\,\omega_1, \omega_2\right) = \gamma_3(z \mid \omega_1, \omega_2),$$

and in accordance with this notation we put

$$\Gamma_2(z \mid \omega_1, \omega_2) = \gamma_0(z \mid \omega_1, \omega_2).$$

These functions evidently correspond to the functions

$$\sigma(z), \; \sigma_1(z), \; \sigma_2(z), \; \sigma_3(z),$$

in WEIERSTRASS' theory of elliptic functions.

Omitting the zero argument, we take

$$\Gamma_2\left(\frac{\omega_1}{2}\,\middle|\,\omega_1, \omega_2\right) = \gamma_1(\omega_1, \omega_2) \text{ and two similar equations,}$$

so that
$$\prod_{r=0}^{2-1} \prod_{s=0}^{2-1}{}' \Gamma_2\left(\frac{r\omega + s\omega_2}{2}\right) = \gamma_1\gamma_2\gamma_3,$$

the parameters ω_1 and ω_2 being omitted.

And now, from the result of the preceding paragraph,

$$\gamma_1\gamma_2\gamma_3 = \rho_2^3(\omega_1, \omega_2)\, 2^{{}_2S_1'(o)-\iota}\; e^{-3 \cdot 2(m+m')\pi\iota_2 S_1'(o)},$$

so that
$$\rho_2(\omega_1, \omega_2) = \sqrt[3]{(\gamma_1\gamma_2\gamma_3)}\, 2^{\frac{1}{3}[1-{}_2S_1'(o)]}\; e^{2(m+m')\pi\iota\,{}_2S_1'(o)}.$$

We thus express the double Stirling function of ω_1 and ω_2 in terms of the product of double gamma functions whose argument is a half quasi-period.

We have previously seen in the theory of the simple gamma function that

$$\rho_1(\omega) = 2^{\frac{1}{2}}\Gamma_1\left(\frac{\omega}{2}\right),$$

and the formula just obtained is the natural extension of this result.

§ 65. From the results of §§ 62 and 63 we see that we may express the multiplication formula for $\Gamma_2(z)$ in the form

$$\Gamma_2(mz) = \frac{\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\Gamma_2\left(z+\frac{r\omega_1+s\omega_2}{m}\right)}{\rho_2^{m^2-1}(\omega_1,\omega_2)} e^{-{}_2S_1'(mz)\log m + (m^2-1)2(m+m')\pi\iota_2 S_1'(o)}.$$

We now proceed to obtain this result at once from the expression of $\log\frac{\Gamma_2(z)}{\rho_2(\omega_1,\omega_2)}$ as a contour integral.

We have seen (§ 45) that, when a is positive with respect to the ω's,

$$\log\frac{\Gamma_2(a)}{\rho_2(\omega_1,\omega_2)} = {}_2S_0(a)(M+m+m')2\pi\iota + {}_2S_1'(o)\,2M\pi\iota + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz,$$

and, therefore, m being a positive integer,

$$\log\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\frac{\Gamma_2\left(a+\frac{r\omega_1+s\omega_2}{m}\right)}{\rho_2(\omega_1,\omega_2)}\right\} = (M+m+m')\,2\pi\iota\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}{}_2S_0\left(a+\frac{r\omega_1+s\omega_2}{m}\right)$$
$$+ m^2{}_2S_1'(o)2M\pi\iota + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}e^{-z\cdot\frac{r\omega_1+s\omega_2}{m}}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}.$$

But
$$1-e^{-z\frac{\omega_1}{m}}+\ldots+e^{-z\frac{(m-1)\omega_1}{m}} = \frac{1-e^{-\omega_1 z}}{1-e^{-\frac{\omega_1 z}{m}}}$$

and (§ 14)

$$\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}{}_2S_0\left(a+\frac{r\omega_1+s\omega_2}{m}\right) = {}_2S_0\left(a\,\middle|\,\frac{\omega_1}{m},\frac{\omega_2}{m}\right) - m^2{}_2B_1(\omega_1,\omega_2) + {}_2B_1\left(\frac{\omega_1}{m},\frac{\omega_2}{m}\right)$$
$$= {}_2S_0(ma\,|\,\omega_1,\omega_2) + (1-m^2)\,{}_2B_1(\omega_1,\omega_2).$$

Therefore when a is positive with respect to the ω's,

$$\log\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\frac{\Gamma_2\left(a+\frac{r\omega_1+s\omega_2}{m}\right)}{\rho_2(\omega_1,\omega_2)}\right\} = (M+m+m')2\pi\iota\,{}_2S_0(ma\,|\,\omega_1,\omega_2) + 2M\pi\iota\,{}_2S_1'(o)$$
$$+ (m+m')2\pi\iota(1-m^2)\,{}_2B_1(\omega_1,\omega_2) + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}dz}{(1-e^{-\frac{\omega_1 z}{m}})(1-e^{-\frac{\omega_2 z}{m}})}.$$

Since m is a positive integer, the axis L defined with reference to the parameters ω_1, ω_2 is the same as that defined with reference to the parameters $m\omega_1$, $m\omega_2$ for the lines representing these two sets of parameters are coincident. If then we change z into mz, the integral last written becomes one which (§ 45) is equal to

$$\log\frac{\Gamma_2(ma)}{\rho_2(\omega_1,\omega_2)} + {}_2S_1'(ma)\{\log m - 2M\pi\iota\} - {}_2S_0(ma)(m+m')2\pi\iota,$$

the arithmetic value of $\log m$ being taken.

And therefore

$$\log \prod_{r=0}^{m=1} \prod_{s=0}^{m=1} \left\{ \frac{\Gamma_2\left(a + \frac{r\omega_1 + s\omega_2}{m}\right)}{\rho_2(\omega_1, \omega_2)} \right\} = (m + m') 2\pi\iota(1 - m^2)_2 S_1'(o) + {}_2S_1'(ma) \log m + \log \frac{\Gamma_2(ma)}{\rho_2(\omega_1, \omega_2)},$$

which is the result required. This result has of course only been proved by means of the contour integral, under the assumption that a is positive with respect to the ω's. To establish it in general we should appeal to the principle of continuity.

§ 66. Before concluding the multiplication theory, we deduce expressions for the values of

$$\sum_{p=0}^{m=1} \sum_{q=0}^{m=1}{}' \psi_2^{(r)} \left(\frac{p\omega_1 + q\omega_2}{m}\right), \text{ where } r = 1, 2, 3, \ldots$$

We recall from § 29, that within a circle of sufficiently small radius surrounding the origin, we have

$$\log \Gamma_2(z) = - \log z - z\gamma_{22} - \frac{z^2\gamma_{21}}{2} - \sum_0^\infty \sum{}' \frac{z^3}{3\Omega^3} + \sum_0^\infty \sum{}' \frac{z^4}{4\Omega^4} - \ldots$$

Again, from the multiplication theorem of the preceding paragraph, we have, by a similar expansion,

$$\begin{aligned} \log \Gamma_2(mz) = &- (m + m') 2\pi\iota(1 - m^2)\, {}_2S_1'(o) + (1 - m^2) \log \rho_2(\omega_1, \omega_2) \\ &- \log m \left[{}_2S_1'(o) + mz\, {}_2S_1^{(2)}(o) + \frac{m^2z^2}{2}\, {}_2S_1^{(3)}(o)\right] + \log \Gamma_2(z) \\ &+ \log \prod_{p=0}^{m-1} \prod_{q=0}^{m-1}{}' \Gamma_2\left(\frac{p\omega_1 + q\omega_2}{m}\right) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \left[z\psi_2^{(1)}\left(\frac{p\omega_1 + q\omega_2}{m}\right)\right. \\ &\left. + \frac{z^2}{2!} \psi_2^{(2)}\left(\frac{p\omega_1 + q\omega_2}{m}\right) + \ldots\right]. \end{aligned}$$

Combine these two theorems, and equate coefficients of various powers of z in the resulting identity. We find

$$\gamma_{22}(\omega_1, \omega_2) = \frac{m}{m-1}\, {}_2S_0'(o) \log m - \frac{1}{m-1} \sum_{p=0}^{m-1} \sum_{q=0}^{m=1}{}' \psi_2^{(1)}\left(\frac{p\omega_1 + q\omega_2}{m}\right),$$

$$\gamma_{21}(\omega_1, \omega_2) = \frac{m^2}{m^2 - 1}\, {}_2S_0^{(2)}(o) \log m - \frac{1}{m^2 - 1} \sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \psi_2^{(2)}\left(\frac{p\omega_1 + q\omega_2}{m}\right)$$

and, when r is greater than 2,

$$\sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \psi_2^{(r)}\left(\frac{p\omega_1 + q\omega_2}{m}\right) = (-)^r (r - 1)! (m^r - 1) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^r}.$$

Transformation Theory.

§ 67. We shall now consider the theory of the transformation of the parameters of the double gamma function. It must not be supposed that we intend to consider the general linear transformation.

There exists no such theory for the present functions—at any rate, no theory having the simplicity and elegance which is characteristic of the elliptic functions, and the reason is obvious—the change of ω_1 into $\omega_1 + \omega_2$ makes no difference of form in such a series as

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3},$$

but it makes a change of comparatively great complexity in such a series as

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3}.$$

The former series is the basis of those occurring in the theory of elliptic functions, the latter of those occurring in double gamma functions. We shall then limit our consideration to transformations which result from the change of ω_1 and ω_2 into ω_1/p and ω_2/q respectively, p and q being positive integers.

By definition we have

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + \Omega)^3}.$$

where $\Omega = m_1\omega_1 + m_2\omega_2$.

Hence
$$\psi_2^{(3)}\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{\left(z + \frac{m_1\omega_1}{p} + \frac{m_2\omega_2}{q}\right)^3}$$
$$= -2 \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \left[\sum_{m_1}^{\infty} \sum_{m_2 \, 0} \frac{1}{\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q} + m_1\omega_1 + m_2\omega_2\right)^3} \right]$$
$$= \sum_{s=0}^{p-1} \sum_{s=0}^{q-1} \psi_2^{(3)}\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right),$$

it being understood that the parameters when not expressed are always ω_1 and ω_2.

On integrating successively three times with respect to z, we shall find

$$\log \Gamma_2 \left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \chi_2(z) + \log \prod_{r=0}^{p-1} \prod_{s=0}^{q-1} \Gamma_2 \left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$

where $\chi_2(z)$ is an algebraical polynomial in z of order 2.

As has been stated, it is possible to obtain $\chi_2(z)$ by purely algebraical processes, use being made of the limit theorems previously established. We may, however, obtain its value as follows :—

In the relation which has been obtained, change z into $z + \omega_1/p$ and subtract the first result from the one so formed. We find by § 23,

$$\log \frac{\Gamma_1\left(z \,\middle|\, \frac{\omega_2}{q}\right)}{\rho_1\left(\frac{\omega_3}{q}\right)} - 2m_{p,q}\,\pi\iota\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right) = \chi_2(z) - \chi_2\left(z + \frac{\omega_1}{p}\right)$$

$$+ \log \prod_{s=0}^{q-1} \left\{ \frac{\Gamma_1\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right)}{\rho_1(\omega_2)} \right\} - 2m\,\pi\iota \sum_{s=0}^{q-1} S_1'\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right),$$

where m has the value asigned in § 22, and where $m_{p,q} = 0$, unless $\left\{\begin{array}{l} \frac{\omega_1}{p} \text{ does} \\ \frac{\omega_1}{p} + \frac{\omega_2}{q} \text{ does not} \end{array}\right.$ lie between -1 and $-\frac{\omega_2}{q}$, in which case $m_{p,q} = \pm 1$ as $I\left(\frac{\omega_2}{q}\right)$ is positive or negative Now ("Theory of the Gamma Function," § 7)

$$\log \prod_{s=0}^{q-1} \Gamma_1\left(z + \frac{s\omega_2}{q}\right) = \log \Gamma_1\left(z \,\middle|\, \frac{\omega_2}{q}\right) + q \log \rho_1(\omega_2) - \log \rho_1\left(\frac{\omega_2}{q}\right),$$

and by § 18 of the same paper,

$$S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right) = \sum_{s=0}^{q-1} S_1\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right).$$

We therefore have

$$\chi_2\left(z + \frac{\omega_1}{p}\right) - \chi_2(z) = 2\pi\iota\,(m_{p,q} - m)\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right).$$

Similarly,

$$\chi_2\left(z + \frac{\omega_2}{q}\right) - \chi_2(z) = 2\pi\iota\,(m_{p,q} - m')\, S_1'\left(z \,\middle|\, \frac{\omega_1}{p}\right),$$

where m' has the value assigned in § 21, and $m'_{p,q}$ differs from it in that $p\omega_2 + q\omega_1$ must in the definition be substituted for $\omega_1 + \omega_2$.

Now we have seen that (§ 22) $m - m' = \pm 1$, the upper or lower sign being taken as $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative or positive; and, since p and q are positive integers, the same is true of $m_{p,q} - m'_{p,q}$.

Thus

$$m_{p,q} - m = m'_{p,q} - m' = \mu_{p,q} \text{ (say)},$$

and now $\chi_2(z)$ satisfies the two difference relations

$$\chi_2\left(z + \frac{\omega_1}{p}\right) - \chi_2(z) = 2\pi\iota\,\mu_{p,q}\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right)$$

$$\chi_2\left(z + \frac{\omega_2}{q}\right) - \chi_2(z) = 2\pi\iota\,\mu_{p,q}\, S_1'\left(z \,\middle|\, \frac{\omega_1}{p}\right);$$

and, therefore, since $\chi_2(z)$ is a quadratic polynomial in z, we must have

$$\chi_2(z) = 2\pi\iota\, \mu_{p,q}\ {}_2S_1'\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + \text{constant}.$$

If we determine the constant by making $z = 0$, we have finally

$$\Gamma_2\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \frac{\prod_{r=0}^{p-1}\prod_{s=0}^{q-1} \Gamma_2\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}{\prod_{r=0}^{p-1}\prod_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)} e^{2\pi\iota\mu_{p,q}\, {}_2S_0\left(\ \middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}.$$

From the values of $m_{p,q}$ and m, it is readily seen that $\mu_{p,q} = 0$, unless the axes to $(\omega_1 + \omega_2)$ and $(q\omega_1 + p\omega_2)$ include the axis of -1, in which case

$$\mu_{p,q} = -1 \text{ if } I(\omega_1 + \omega_2) \text{ is } +ve \text{ and } I(q\omega_1 + p\omega_2) -ve$$

$$\mu_{p,q} = +1 \text{ if } I(\omega_1 + \omega_2) \text{ is } -ve \text{ and } I(q\omega_1 + p\omega_2) +ve.$$

§ 68. The constant which enters into the transformation formula of the preceding paragraph can be expressed in terms of $\rho_2(\omega_1, \omega_2)$ and $\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$. For this purpose we consider the contour integral which represents the double gamma function.

Since p is a positive integer,

$$\frac{1 - e^{-\omega_1 a}}{1 - e^{-\frac{\omega_1 a}{p}}} = \sum_{r=0}^{p-1} e^{-a\frac{r\omega_1}{p}}.$$

And therefore if the integral, its contour, and the logarithm which occurs in the subject of integration, be defined as in Part III., we have

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}\, dz}{\left(1 - e^{-\frac{\omega_1 z}{p}}\right)\left(1 - e^{-\frac{\omega_2 z}{q}}\right)}$$

$$= \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}\sum_{r=0}^{p-1}\sum_{s=0}^{q-1} e^{-z\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz$$

for the bisector of the angle between the axes of ω_1/p and ω_2/q is the same line as the bisector of the angle between the axes of $1/\omega_1$ and $1/\omega_2$. We therefore have by § 45, when a is positive with respect to the ω's,

$$\log \frac{\Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}{\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)} - {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)(m_{p,q} + m'_{p,q})\, 2\pi\iota - {}_2S_1'\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\mu_{p,q}\pi\iota$$

$$= \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} \log\left\{\frac{\Gamma_2\left(a + \frac{r\omega_1}{p}, \frac{s\omega_2}{q}\right)}{\rho_2(\omega_1, \omega_2)}\right\} - (m + m')\, 2\pi\iota \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} {}_2S_0\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$

$$- \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} {}_2S_1'\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) 2M\pi\iota.$$

Now by § 39 $M = 0$, unless the axes of $\frac{1}{L}$ and $(\omega_1 + \omega_2)$ include the axis of -1, in which case

$$M = \mp 1 \text{ as } I(\omega_1 + \omega_2) \text{ is positive or negative.}$$

Therefore $M_{p,q} = 0$, unless the axes of $\frac{1}{L}$ and $(q\omega_1 + p\omega_2)$ include the axis of -1, in which case

$$M_{p,q} = \mp 1, \text{ as } I(q\omega_1 + p\omega_2) \text{ is positive or negative.}$$

Again, by § 14,

$$ {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} {}_2S_0\left(a + \frac{r\omega_1}{p} + \frac{\omega_2}{q}\right) + pq\,{}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right).$$

We therefore have

$$\log \frac{\Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}{\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)} = \sum_{r=0}^{p-1}\sum_{r=0}^{q-1} \log \frac{\Gamma_2\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}{\rho_2(\omega_1, \omega_2)}$$
$$+ \left[pq\,{}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right](m + m')\,2\pi\iota + {}_2S'_1\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\,(M_{p,q} - M)$$
$$+ {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\,[m_{p,q} + m'_{p,q} + M_{p,q} - m - m' - M].$$

But from the values which have just been given, it is clear that

$$M_{p,q} - M = -\mu_{p,q}.$$

We thus have, when a is positive with respect to the ω's,

$$\log \Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} \log \Gamma_2\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) + 2\pi\iota\mu_{p,q}\,{}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$$
$$+ \log \rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) - pq \log \rho_2(\omega_1, \omega_2) + \left[pq\,{}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right](m + m')\,2\pi\iota$$
$$- {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\mu_{p,q}.$$

This result agrees with that of § 67, and on comparison of the absolute term we see that

$$\log \prod_{r=0}^{p-1}\prod_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) = pq\,\{\log \rho_2(\omega_1, \omega_2) - (m + m')\,2\pi\iota\,{}_2B_1(\omega_1, \omega_2)\}$$
$$- \left\{\log \rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) - (m + m' + \mu_{p,q})\,2\pi\iota\,{}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right\}.$$

This may be regarded as the transformation formula for the double Stirling function $\rho_2(\omega_1, \omega_2)$.

Notice that when $p = q = m$, $\mu_{p,q} = 0$, ${}_2B_1\left(\frac{\omega_1}{m'}, \frac{\omega_2}{m}\right) = {}_2B_1(\omega_1, \omega_2)$, and therefore the preceding formula becomes

$$\log \prod_{r=0}^{p-1} \prod_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1 + s\omega_2}{m}\right) = m^2 \log \rho_2(\omega_1, \omega_2) - \log \rho_2\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right)$$
$$+ (1 - m^2)\, 2\pi\iota (m + m')\, {}_2B_1(\omega_1, \omega_2).$$

Comparing this with the result obtained in § 63, we find

$$\log \rho_2\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right) = \log \rho_2(\omega_1, \omega_2) + [1 - {}_2S_1'(o)] \log m.$$

This result may also be obtained by the transformation of the line integral which expresses $\log \rho_2(\omega_1, \omega_2)$.

§ 69. We have still to consider the transformation of the first and second double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2) \quad \text{and} \quad \gamma_{22}(\omega_1, \omega_2).$$

With this object we write symbolically

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{\left(m_1 \frac{\omega_1}{p} + m_2 \frac{\omega_2}{q}\right)^m} = \frac{1}{v^m},$$

and we write

$$S_m = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \psi_2^{(m)}\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right), \text{ where } m \geqq 1;$$

so that S_m is a form analogous to the modular forms introduced into the theory of elliptic functions by ABEL.

By § 29 we know that within a circle of sufficiently small radius there exists the expansion

$$\log \Gamma_2(z) = -\log z - z\gamma_{22} - \frac{z^2}{2}\gamma_{21} - \sum_0^{\infty}\sum{}' \frac{z^3}{3\Omega^3} + \sum_0^{\infty}\sum{}' \frac{z^4}{4\Omega^4} - \ldots$$

Take now the formula of § 67

$$\log \Gamma_2\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \log \Gamma_2\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \log \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$
$$+ 2\pi\iota\mu_{p,q}\, {}_2S_0\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right),$$

and expand in powers of z.

We evidently obtain

$$-\log z - \gamma_{22}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)z - \gamma_{21}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\frac{z^2}{2} - \frac{z^3}{3v^3} + \frac{z^4}{4v^4} - \quad . \; .$$

$$= -\log z - \gamma_{22}(\omega_1, \omega_2)z - \gamma_{21}(\omega_1 \;\omega_2)\frac{z^2}{2} - \frac{z^3}{3\omega^3} + \frac{z^4}{4\omega^4} - \ldots$$

$$+ z\mathrm{S}_1 + \frac{z^2}{2!}\mathrm{S}_2 + \ldots + 2\pi\iota\mu_{p,q}\left[{}_2\mathrm{S}_0\left(o \,\Big|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + z\,{}_2\mathrm{S}_0'\left(o \,\Big|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right.$$

$$\left. + \frac{z^2}{2}\,{}_2\mathrm{S}_0^{(2)}\left(o \,\Big|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + \ldots\right],$$

where we put symbolically

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^m} = \frac{1}{\omega^m}.$$

And now, equating coefficients of the various powers of z,

$$\gamma_{22}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \gamma_{22}(\omega_1, \omega_2) - \sum_{r=0}^{p-1}\sum_{s=0}^{q-1}{}' \psi_2'\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - 2\pi\iota\mu_{p,q}\,{}_2\mathrm{S}_0'\left(o \,\Big|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$$

$$\gamma_{21}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \gamma_{21}(\omega_1, \omega_2) - \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} \psi_2^{(2)}\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - 2\pi\iota\mu_{p,q}\,{}_2\mathrm{S}_0^{(2)}\left(o \,\Big|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right),$$

the transformation equations for the first and second gamma modular forms.

Note that we also have, where $m > 2$,

$$\sum_{r=0}^{p-1}\sum_{s=0}^{q-1}{}' \psi_2^{(m)}\left(\frac{r\omega_1}{p}, \frac{s\omega_2}{q}\right) = (-)^m(m-1)!\left\{\frac{1}{v^m} - \frac{1}{\omega^m}\right\}.$$

If $\tau = \omega_2/\omega_1$ we may put $\omega^2\,\gamma_{21}(\omega_1, \omega_2) = g_{21}(\tau)$; and now, putting $p = 1$, $q = n$, we have

$$g_{21}\left(\frac{\tau}{n}\right) - g_{21}(\tau) = -\,\omega_1^2 \sum_{s=1}^{n-1} \psi_2^{(2)}\left(\frac{s\omega_2}{n}\right) - 2\pi\iota\mu_{1,n}\,\frac{n}{\tau}.$$

And putting $p = n$, $q = 1$,

$$n^2 g_{21}(n\tau) - g_{21}(\tau) = -\,\omega_1^2 \sum_{r=1}^{n-1} \psi_2^{(2)}\left(\frac{r\omega_1}{n}\right) - 2\pi\iota\mu_{n,1}\,\frac{n}{\tau}.$$

We get analogous formulæ by writing $\omega_1\,\gamma_{22}(\omega_1, \omega_2) = g_{22}(\tau)$.

The analogy between these results and those obtained in the analysis of elliptic functions is obvious. We cannot obtain, however, results which will connect such expressions as

$$g_{21}(\tau + 1) \text{ and } g_{21}(\tau).$$

Integral Formulæ.

§ 70. In the general theory of multiple gamma functions the fundamental integral formula expresses the fact that the integral of the n-ple gamma function can be expressed in terms of $(n+1)$-ple gamma functions of specialised parameters. As we have not yet defined the treble gamma function, we cannot prove this theorem for the case when $n = 2$. In the case when $n = 1$, this proposition reduces to ALEXEIEWSKY's theorem ("Theory of the G Function," § 13). We proceed first to translate this theorem into the notation of the present paper, and then to give an alternative proof capable of extension to the n-ple gamma functions.

The G function is substantially the double gamma function with equal parameters, the two being connected by the relation

$$\Gamma_2^{-1}(z \mid \omega, \omega) = \mathrm{G}\left(\frac{z}{\omega}\right)(2\pi)^{-\frac{z}{2\omega}} \omega^{z \cdot \frac{z-2\omega}{2\omega^2}+1}.$$

["Theory of the G Function," § 29.]

By differentiating ALEXEIEWSKY's theorem we obtain

$$o = \tfrac{1}{2}\log 2\pi - z - a + \tfrac{1}{2} + (z + a - 1)\frac{d}{dz}\log\Gamma(z+a) - \frac{d}{dz}\log\mathrm{G}(z+a),$$

and therefore, writing z for $z + a$, and substituting from the relation just quoted, we find

$$o = -\frac{z}{\omega} + \tfrac{1}{2} + \left(\frac{z}{\omega} - 1\right)\omega\frac{d}{dz}\log\Gamma\left(\frac{z}{\omega}\right) + \omega\frac{d}{dz}\log\Gamma_2(z \mid \omega) + \frac{z-\omega}{\omega}\log\omega.$$

But $$\log\Gamma\left(\frac{z}{\omega}\right) = \log\Gamma_1(z \mid \omega) - \left(\frac{z}{\omega} - 1\right)\log\omega.$$

We thus have $$z\,\psi_1'(z \mid \omega) + \omega\,\psi_2'(z + \omega \mid \omega) = \mathrm{S}_1'(z \mid \omega). \qquad \text{(i.)}$$

On integration we have

$$\int_0^a \log\Gamma_1(a \mid \omega)\,da = a\log\Gamma_1(a \mid \omega) - \mathrm{S}_1(a \mid \omega) + \omega\log\frac{\Gamma_2(a+\omega \mid \omega)}{\Gamma_2(\omega \mid \omega)}.$$

We may put $z + a$ in place of z, so that

$$(z+a)\frac{d}{dz}\log\Gamma_1(z+a \mid \omega) + \omega\frac{d}{dz}\log\Gamma_2(z+a+\omega \mid \omega) = \mathrm{S}_1'(z+a \mid \omega).$$

And now, on integration with respect to z between the limits 0 and z, we obtain the extended formula

$$\int_0^z \log\Gamma_1(z+a \mid \omega)\,dz = (z+a)\log\Gamma_1(z+a \mid \omega) + \omega\log\frac{\Gamma_2(z+a+\omega \mid \omega)}{\Gamma_2(a+\omega \mid \omega)}$$
$$- \mathrm{S}_1(z+a \mid \omega) + \mathrm{S}_1(a \mid \omega) - a\log\Gamma_1(a \mid \omega).$$

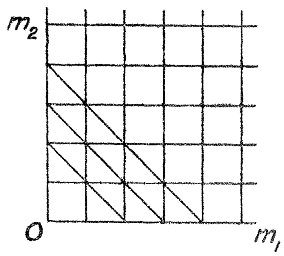
§ 71. There are two alternative methods of obtaining this formula, or rather of obtaining the fundamental relation (1).

Firstly, we may directly transform the series which expresses $\psi_2'(z \mid \omega, \omega)$.

We have, from the definition,

$$- \psi_2'(z \mid \omega, \omega) = z\gamma_{21}(\omega, \omega) + \gamma_{22}(\omega, \omega) + \frac{1}{z}$$
$$+ \sum_{0}^{\infty}\sum_{m_1\ m_2}{}' \left[\frac{1}{z + (m_1 + m_2)\omega} - \frac{1}{(m_1 + m_2)\omega} + \frac{z}{(m_1 + m_2)^2\omega^2}\right].$$

Put now $m_1 + m_2 = \epsilon$, a change which is equivalent to grouping together terms corresponding to points on the cross lines of the figure.



There we have

$$- \psi_2^1(z \mid \omega) = \gamma_{22}(\omega) + z\gamma_{21}(\omega) + \frac{1}{z} + \sum_{\epsilon=1}^{\infty}\left[\frac{\epsilon + 1}{z + \epsilon\omega} - \frac{\epsilon + 1}{\epsilon\omega} + \frac{(\epsilon + 1)z}{\epsilon^2\omega^2}\right]$$
$$= \gamma_{22}(\omega) + z\gamma_{21}(\omega) + \frac{1}{z} + \sum_{\epsilon=1}^{\infty}\left[\frac{1}{z + \epsilon\omega} - \frac{1}{\epsilon\omega}\right]\left(1 - \frac{z}{\omega}\right) + \frac{z}{\omega^2}\cdot\frac{\pi^2}{6}.$$

Now we have seen in § 28 that

$$\gamma_{21}(\omega) = \frac{1}{\omega^2}\left[\log\omega - 1 - \gamma - \frac{\pi^2}{6}\right].$$
$$\gamma_{22}(\omega) = \frac{1}{\omega}\left[\gamma - \tfrac{1}{2} - \log\omega\right].$$

We therefore have

$$- \psi_2'(z \mid \omega) = \left(\frac{z}{\omega} - 1\right)\psi_1'(z - \omega \mid \omega) - \frac{1}{\omega}S_1'(z - \omega \mid \omega),$$

which is equivalent to the former relation.

Secondly, we may make use of the contour-integral expression in the following manner.

We have ("Theory of the Gamma Function," § 30), when a is positive with respect to ω,

$$\psi_1^{(2)}(a \mid \omega) = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-az}(-z)\{\log(-z) + \gamma\}}{1 - e^{-\omega z}}\,dz.$$

Therefore, under the same limitation,

$$\frac{\partial}{\partial\omega}\psi_1^{(2)}(a\,|\,\omega) = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}}\frac{e^{-(a+\omega)z}(-z)^2\{\log(-z)+\gamma\}}{(1-e^{-\omega z})^2}\,dz = \psi_2^{(3)}(a+\omega\,|\,\omega).$$

Now, since

$$\psi_1^{(2)}(a\,|\,\omega) = \sum_{m=0}^{\infty}\frac{1}{(a+m\omega)^2},$$

it is evident that $\psi_1^{(2)}(a\,|\,\omega)$ is homogeneous of degree -2 in a and ω.

Therefore, by EULER'S theorem,

$$\left(a\frac{\partial}{\partial a} + \omega\frac{\partial}{\partial\omega}\right)\psi_1^{(2)}(a\,|\,\omega) = -2\psi_1^{(2)}(a\,|\,\omega),$$

so that

$$a\psi_1^{(3)}(a\,|\,\omega) + \omega\psi_2^{(3)}(a+\omega\,|\,\omega) = 2\psi_1^{(2)}(a\,|\,\omega).$$

On integrating twice with respect to a, we obtain

$$a\psi_1'(a\,|\,\omega) + \omega\psi_2'(a+\omega\,|\,\omega) = \chi_1(a\,|\,\omega),$$

where $\chi_1(a\,|\,\omega)$ is a linear function of a.

Changing a into $a+\omega$, and subtracting, we see that $\chi_1(a\,|\,\omega)$ satisfies the difference equation characteristic of $S_1'(a\,|\,\omega)$, so that it can only differ from this function by a constant, which will vanish, as we see by making $a = 0$.

We therefore obtain again the relation required.

§ 72. We proceed now to the analogues of RAABE'S formula. This formula may be written ("Theory of the Gamma Function," § 8)

$$\int_0^{\omega}\log\Gamma_1(z+a\,|\,\omega)\,dz = a\log a - a + \frac{\omega}{2}\log\frac{2\pi}{\omega}.$$

We will evaluate

$$\int_0^{\omega_1}\log\Gamma_2(z+a\,|\,\omega_1,\omega_2)\,dz \quad\text{and}\quad \int_0^{\omega_2}\log\Gamma_2(z+a\,|\,\omega_1\,\omega_2)\,dz.$$

The method which will be employed is the same as that by which RAABE'S formula itself was originally obtained; it was, in fact, first invented for the proof of the present theorem.

Let

$$f(a) = \int_0^{\omega_1}\log\Gamma_2(z+a\,|\,\omega_1,\omega_2)\,dz.$$

Then

$$\begin{aligned}\frac{df(a)}{da} &= \int_0^{\omega_1}\frac{\Gamma_2'(z+a)}{\Gamma_2(z+a)}\,dz \\ &= \log\frac{\Gamma_2(a+\omega_1)}{\Gamma_2(a)} \\ &= -\log\Gamma_1(a\,|\,\omega) + \log\rho_1(\omega_2) + 2m\pi\iota\,S_1'(a\,|\,\omega_2),\end{aligned}$$

and, therefore, on integration,

$$\int_0^{\omega_1} \log \Gamma_2 (z + a \mid \omega_1, \omega_2) dz = - \int_0^{a} \log \Gamma_1 (a \mid \omega_2)\, da + a \log \rho_1 (\omega_2)$$

$$+ 2m\pi\iota\, S_1 (a \mid \omega_2) + \int_0^{\omega_1} \log \Gamma_2 (z \mid \omega_1, \omega_2)\, dz.$$

§ 73. We proceed to evaluate the constant term

$$\int_0^{\omega_1} \log \Gamma_2 (z \mid \omega_1, \omega_2)\, dz,$$

by an application of the multiplication theory for the case $m = 2$.

We have

$$\int_0^{\omega_1} \log \Gamma_2 (2z)\, dz = \tfrac{1}{2} \int_0^{2\omega_1} \log \Gamma_2 (z)\, dz,$$

and therefore, by § 65,

$$\int_0^{\omega_1} dz \left\{ \log \Gamma_2 (z) + \log \Gamma_2 \left(z + \frac{\omega_1}{2} \right) + \log \Gamma_2 \left(z + \frac{\omega_2}{2} \right) + \log \Gamma_2 \left(z + \frac{\omega_1 + \omega_2}{2} \right) \right.$$

$$\left. - 3 \log \rho_2 (\omega_1, \omega_2) - {}_2S_1' (2z) \log 2 + 3 (m + m') 2\pi\iota\, {}_2S_1' (o) \right\}$$

$$= \tfrac{1}{2} \int_0^{\omega_1} dz \{ 2 \log \Gamma_2(z) - \log \Gamma_1 (z \mid \omega_2) + \log \rho_1 (\omega_2) + 2m\pi\iota\, S_1' (z \mid \omega_2) \}.$$

Put $a = \frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2}(\omega_1 + \omega_2)$ successively in the formula which we have just obtained for $\int_0^{\omega_1} \log \Gamma_2 (z + a)\, dz$, and substitute in the formula just written. We obtain

$$3 \int_0^{\omega_1} \log \Gamma_2 (z) dz$$

$$= \left[\int_0^{\frac{\omega_1}{2}} + \int_0^{\frac{\omega_2}{2}} + \int_0^{\frac{\omega_1 + \omega_2}{2}} - \tfrac{1}{2} \int_0^{\omega_1} \right] \{ \log \Gamma_1 (z \mid \omega_2) dz \} - \left(\frac{\omega_1}{2} + \omega_2 \right) \log \rho_1 (\omega_2)$$

$$+ 2m\pi\iota \left[\tfrac{1}{2} S_1 (\omega_1 \mid \omega_2) - S_1 \left(\frac{\omega_1}{2} \mid \omega_2 \right) - S_1 \left(\frac{\omega_2}{2} \mid \omega_2 \right) - S_1 \left(\frac{\omega_1 + \omega_2}{2} \mid \omega_2 \right) \right]$$

$$+ 3\omega_1 \log \rho_2 (\omega_1, \omega_2) + \tfrac{1}{2}\, {}_2S_1 (2\omega_1) \log 2 - 3\omega_1 (m + m') 2\pi\iota\, {}_2S_1' (o).$$

By means of the formula

$$\int_0^{a} \log \Gamma_1 (a \mid \omega)\, da = a \log \Gamma_1 (a \mid \omega) - S_1 (a \mid \omega) + \omega \log \frac{\Gamma_1 (a + \omega \mid \omega)}{\rho_1 (\omega)},$$

we see that

$$\left[\int_0^{\frac{\omega_1}{2}} + \int_0^{\frac{\omega_2}{2}} + \int_0^{\frac{\omega_1+\omega_2}{2}} - \tfrac{1}{2}\int_0^{\omega_1}\right] \log \Gamma_1(z \mid \omega_2)dz$$

$$= \frac{\omega_1}{2}\log \Gamma_1\left(\frac{\omega_1}{2}\mid\omega_2\right) + \frac{\omega_2}{2}\log \Gamma_1\left(\frac{\omega_2}{2}\mid\omega_2\right) + \frac{\omega_1+\omega_2}{2}\log \Gamma_1\left(\frac{\omega_1+\omega_2}{2}\mid\omega_2\right)$$

$$- \frac{\omega_1}{2}\log \Gamma_1(\omega_1\mid\omega_2) + \tfrac{1}{2}S_1(\omega_1\mid\omega_2) - S_1\left(\frac{\omega_1}{2}\mid\omega_2\right) - S_1\left(\frac{\omega_2}{2}\mid\omega_2\right) - S_1\left(\frac{\omega_1+\omega_2}{2}\mid\omega_2\right)$$

$$+ \omega_2 \log\left\{\frac{\Gamma_2(\omega_2+\frac{\omega_1}{2}\mid\omega_2)\Gamma_2(\omega_2+\frac{\omega_2}{2}\mid\omega_2)\Gamma_2(\omega_2+\frac{\omega_1+\omega_2}{2}\mid\omega_2)}{\Gamma_1^{\frac{1}{2}}(\omega_1+\omega_2\mid\omega_2)\,\rho_1^{\frac{5}{2}}(\omega_2)}\right\},$$

and this expression in turn, by utilising the multiplication formulæ when $m = 2$ for the simple and double gamma functions and the simple Bernoullian function, is equal to

$$\left(-2\omega_2+\frac{\omega_1}{2}\right)\log\rho_1(\omega_2) + \left[\frac{\omega_1}{4} - \frac{\omega_2}{4} - \frac{\omega_1^2}{2\omega_2} + \frac{\omega_2}{2}\,{}_2S_1'(\omega_1+\omega_2\mid\omega_2,\omega_2)\right]\log 2$$

$$+ \tfrac{3}{4}S_2'(o\mid\omega_2) - S_1\left(\frac{\omega_2}{2}\mid\omega_2\right) + \frac{\omega_2}{2}\log\Gamma_1\left(\frac{\omega_1+\omega_2}{2}\mid\omega_2\right)$$

$$+ \frac{\omega_2}{2}\log\left\{\frac{\Gamma_2^2\left(\frac{\omega_2}{2}\mid\omega_2\right)\rho_1^3(\omega_2)}{\Gamma_1\left(\frac{\omega_1+\omega_2}{2}\mid\omega_2\right)\Gamma_1^2\left(\frac{\omega_2}{2}\mid\omega_2\right)}\right\} + \frac{3\omega_2}{2}\log\rho_2(\omega_2,\omega_2).$$

Now, on making $\omega_1 = \omega_2$, we obtain from the multiplication formula

$$\log\left[\Gamma_2^2\left(\frac{\omega_2}{2}\mid\omega_2\right)\rho_1(\omega_2)\right] = 3\log\rho_2(\omega_2,\omega_2) + [{}_2S_1'(o\mid\omega_2) - 1]\log 2.$$

Therefore the expression which we have just found reduces to

$$\left(-2\omega_2+\frac{\omega_2}{2}\right)\log\rho_1(\omega_2) + \left\{\frac{\omega_1}{4} - \frac{\omega_2}{4} - \frac{\omega_1^2}{2\omega_2} + \frac{\omega_2}{2}\,{}_2S_1'(\omega_1+\omega_2\mid\omega_2,\omega_2)\right.$$

$$\left. + \frac{\omega_2}{2}\,{}_2S_1'(o\mid\omega_2,\omega_2)\right\}\log 2$$

$$+ \tfrac{3}{4}S_2'(o\mid\omega_2) - S_1\left(\frac{\omega_2}{2}\mid\omega_2\right) + 3\omega_2\log\rho_2(\omega_2,\omega_2).$$

And therefore

$$3\int_0^{\omega_1}\log\Gamma_2(z\mid\omega_1,\omega_2)dz$$

$$= 3\omega_1\log\rho_2(\omega_1,\omega_2) + 3\omega_2\log\rho_2(\omega_2,\omega_2) - 3\omega_2\log\rho_1(\omega_2)$$

$$- 3\omega_1(m+m')2\pi\iota\,{}_2S_1(o) + (1+2m\pi\iota)\left[-S_1\left(\frac{\omega_2}{2}\mid\omega_2\right) + \tfrac{3}{4}S_2'(o\mid\omega_2)\right],$$

since the coefficient of log 2 is equal to

$$-\frac{\omega_1^2}{2\omega_2}+\frac{\omega_1}{4}-\frac{\omega_2}{2}+\frac{\omega_2}{2}\,{}_2S_1'(\omega_1+\omega_2\,|\,\omega_2,\omega_2)+\frac{\omega_2}{2}\,{}_2S_1'(o\,|\,\omega_2,\omega_2)+\tfrac{1}{2}\,{}_2S_1(2\omega_1\,|\,\omega_1,\omega_2)=0.$$

We have, then, finally,

$$\int_0^{\omega_1}\log\Gamma_2(z)dz=\omega_1[\log\rho_2(\omega_1,\omega_2)-(m+m')2\pi\iota\,{}_2S_1'(o)]+\omega_2\log\frac{\rho_2(\omega_2,\omega_2)}{\rho_1(\omega_2)}+\frac{\omega_2}{12}(1+2m\pi\iota).$$

We thus see that substantially the double Stirling function of ω_1 and ω_2 is expressed by $\frac{1}{\omega_1}\int_0^{\omega_1}\log\Gamma_2(z)\,dz$, or, by symmetry, by $\frac{1}{\omega_2}\int_0^{\omega_2}\log\Gamma_2(z)\,dz$. We have, in fact, the relation

$$\rho_2(\omega_1,\omega_2)-(m+m')2\pi\iota\,{}_2S_1'(o)=\frac{1}{\omega_1}\int_0^{\omega_1}\log\Gamma_2(z)dz-\frac{\omega_2}{\omega_1}\log A+\frac{\omega_2}{12\omega_1}[\log\omega_2-2m\pi\iota],$$

where A is the Glaisher-Kinkelin constant ("Theory of the G Function," § 3); and therefore

$$\rho_2(\omega_1,\omega_2)-(m+m')\,2\pi\iota\,{}_2S_1'(o)$$
$$=\frac{\omega_1}{\omega_1^2-\omega_2^2}\int_0^{\omega_1}\log\Gamma_2(z)\,dz-\frac{\omega_2}{\omega_1^2-\omega_2^2}\int_0^{\omega_2}\log\Gamma_2(z)\,dz+\frac{\omega_1\omega_2}{12(\omega_1^2-\omega_2^2)}\log\frac{\omega_2}{\omega_1};$$

for, by § 22, $\log\omega_2-\log\omega_1-2(m-m')\pi\iota=\log\frac{\omega_2}{\omega_1}$.

§ 74. We may readily prove these results by the relation which exists between A and $\rho_2(\omega)$. [The latter is a convenient way of writing $\rho_2(\omega,\omega)$.]

For, when each of the parameters is equal to ω, we have ("Theory of the G Function," § 29)

$$\Gamma_2^{-1}(z\,|\,\omega)=G\left(\frac{z}{\omega}\right)(2\pi)^{-\frac{z}{2\omega}}\,\omega^{\frac{(z-\omega)^2}{2\omega^2}+\frac{1}{2}},$$

and therefore

$$G(\tfrac{1}{2})=\Gamma_2^{-1}\left(\frac{\omega}{2}\,|\,\omega\right)(2\pi)^{\frac{1}{4}}\,\omega^{-\frac{5}{8}}.$$

Now from the multiplication theorem for the double gamma functions, when $m=2$, we have

$$\Gamma_2^2\left(\frac{\omega}{2}\,|\,\omega\right)\left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}}=\rho^3(\omega)\,2^{-\frac{7}{12}}.$$

Hence

$$G(\tfrac{1}{2})=\frac{(2\pi)^{\frac{1}{2}}}{\omega^{\frac{7}{8}}}\,\rho_2^{-\frac{3}{2}}(\omega)\,2^{\frac{7}{24}}.$$

Now it has been seen that ("Theory of the G Function," § 17)

$$\mathrm{A} = \frac{2^{\frac{1}{36}} e^{\frac{1}{12}}}{\pi^{\frac{1}{6}} \mathrm{G}^{\frac{2}{3}}(\frac{1}{2})},$$

and therefore

$$\mathrm{A} = \frac{\rho_2(\omega)}{\rho_1(\omega)} (e\omega)^{\frac{1}{12}},$$

which is the relation which was used at the end of the preceding paragraph.

§ 75. It is interesting as a verification of the algebra to notice that ALEXEIEWSKY's analogue of RAABE's formula ("Theory of the G Function," § 16) yields the result of § 73 in the case where the parameters are equal to one another.

This theorem is expressed by the formula

$$\int_0^1 \log \mathrm{G}(z+1)\, dz = \tfrac{4}{3} \log \mathrm{G}(\tfrac{1}{2}) + \tfrac{7}{12} \log \pi + \tfrac{7}{36} \log 2 - \tfrac{1}{12},$$

and therefore, utilising the relation between the G and double gamma functions,

$$\int_0^\omega \log \Gamma_2(a + \omega \mid \omega)\, da = -\frac{4\omega}{3} \log \mathrm{G}(\tfrac{1}{2}) + \frac{\omega}{6} \log 2\pi + \frac{7\omega}{18} \log 2 + \frac{\omega}{12} - \frac{2\omega}{3} \log \omega.$$

If now we express $\log \mathrm{G}(\frac{1}{2})$ in terms of $\log \rho_2(\omega)$ by the formula of § 74, we find

$$\int_0^\omega \log \Gamma_2(a + \omega \mid \omega)\, da = \omega \log \rho_2(\omega) + \omega \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12},$$

a formula which is equivalent to the result of § 73.

§ 76. By combining the results of §§ 72 and 73, we may now write down the value of

$$\int_0^{\omega_1} \log \Gamma_2(z + a \mid \omega_1, \omega_2)\, dz.$$

For we have seen in § 72 that this integral is equal to

$$-\int_0^a \log \Gamma_1(a \mid \omega_2) + a \log \rho_1(\omega_2) + 2m\pi\iota\, \mathrm{S}_1(a \mid \omega_2) + \int_0^{\omega_1} \log \Gamma_2(z \mid \omega_1, \omega_2)\, dz,$$

which expression in turn is equal to

$$\begin{aligned} &- a \log \Gamma_1(a \mid \omega_2) + \mathrm{S}_1(a \mid \omega_2)[1 + 2m\pi\iota] - \omega_2 \log \Gamma_2(a + \omega_2 \mid \omega_2) + (a + \omega_2) \log \rho_1(\omega_2) \\ &\quad + \omega_1 [\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota\, {}_2\mathrm{S}_1'(o)] + \omega_2 \log \frac{\rho_2(\omega_2, \omega_2)}{\rho_1(\omega_2)} + \frac{\omega_2}{12}(1 + 2m\pi\iota). \end{aligned}$$

Thus

$$\begin{aligned} &\int_0^{\omega_1} \log \Gamma_2(z + a \mid \omega_1, \omega_2)\, dz \\ &\quad = - a \log \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} + \omega_1 [\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota_2 \mathrm{S}_1'(o)] \\ &\quad + (1 + 2m\pi\iota) \left\{ \mathrm{S}_1(a \mid \omega_2) + \frac{\omega_2}{12} \right\} - \omega_2 \log \frac{\Gamma_2(a + \omega_2 \mid \omega_2)}{\rho_2(\omega_2, \omega_2)} \end{aligned}$$

Since, by § 64,

$$\log \rho_2(\omega_1, \omega_2) - (m + m') 2\pi\iota\, {}_2S_1'(o) = \tfrac{1}{3} \log \gamma_1\gamma_2\gamma_3 + \tfrac{1}{3}[1 - {}_2S_1'(o)] \log 2,$$

we see that this formula may equally be written

$$\int_0^{\omega_1} \log \Gamma_2(z + a \,|\, \omega_1, \omega_2)\, dz$$

$$= - a \log \frac{\Gamma_1(a \,|\, \omega_2)}{\rho_1(\omega_2)} + \frac{\omega_1}{3} \log \gamma_1\gamma_2\gamma_3 + \frac{\omega_1}{3}[1 - {}_2S_1'(o)] \log 2$$

$$+ (1 + 2m\pi\iota)\left\{S_1(a \,|\, \omega_2) + \frac{\omega_2}{12}\right\} - \omega_2 \log \frac{\Gamma_2(a + \omega_2 \,|\, \omega_2)}{\rho_2(\omega_2)}.$$

This and the corresponding formula, obtained by the interchange of ω_1 and ω_2, m and m', are the analogues for double gamma functions of RAABE's formula for simple gamma functions.

§ 77. In the particular case when a is positive with respect to the ω's, it is possible to obtain more simply the value of

$$\int_0^{\omega_1} \log \Gamma_2(a \,|\, \omega_1, \omega_2)\, da$$

by means of the contour integrals investigated in Part III.

We give this method of proof as it leads incidentally to an expression as a contour integral for $\log \dfrac{\rho_2(\omega_2)}{\rho_1(\omega_2)}$.

We find, on integrating the expression for $\log \dfrac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}$, given in § 45,

$$\int_0^{\omega_1} \log \Gamma_2(a)\, da - \omega_1 \log \rho_2(\omega_1, \omega_2)$$

$$= (M + m + m') 2\pi\iota \int_0^{\omega_1} {}_2S_0(a)\, da + \omega_1\, {}_2S_1'(o)\, 2M\pi\iota - \frac{\iota}{2\pi} \int_L \frac{(-z)^{-2}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_2 z})}\, dz,$$

and the right-hand side, by an application of the formulæ of §§ 6 and 44, becomes

$$(M + m + m') 2\pi\iota \left[- \omega_1\, {}_2S_1'(o) + \frac{S_2'(o \,|\, \omega_2)}{2}\right] + \omega_1\, {}_2S_1'(o)\, 2M\pi\iota$$

$$- \frac{\iota}{2\pi} \int_{\frac{1}{\omega_2}} \frac{(-z)^{-2}\{\log(-z) - (M + m') 2\pi\iota + \gamma\}}{(1 - e^{-\omega_2 z})}\, dz.$$

Now, reducing the contour to a small circle round the origin, we see that

$$\frac{\iota}{2\pi}(M + m') 2\pi\iota \int_{\frac{1}{\omega_2}} \frac{(-z)^{-2} dz}{1 - e^{-\omega_2 z}} = - \frac{S_2'(o \,|\, \omega_2)}{2}(M + m') 2\pi\iota,$$

and therefore

$$\int_0^{\omega_1} \log \Gamma_2(a)\, da - \omega_1 \log \rho_2(\omega_1, \omega_2) + \omega_1 (m + m') 2\pi\iota\, {}_2S_1'(o) - m 2\pi\iota \frac{S_2'(o\,|\,\omega_2)}{2}$$
$$= -\frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} \frac{(-z)^{-2}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\, dz.$$

Since $\dfrac{S_2'(o\,|\,\omega_2)}{2} = \dfrac{\omega_2}{12}$, we see that, to establish the formula of § 73, it is necessary to show that

$$-\frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{(-z)^{-2}\{\log(-z)+\gamma\}}{1-e^{-\omega z}}\, dz = \omega \log\frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12}.$$

This may be readily done as follows :—

We have

$$\log\frac{\Gamma_2(a\,|\,\omega)}{\rho_2(\omega)} = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega z})^2}\, dz.$$

Integrate with respect to a between ω and 2ω, and we find

$$\int_{\omega}^{2\omega} \log \Gamma_2(a\,|\,\omega)\, da - \omega \log \rho_2(\omega) = -\frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-\omega z}(-z)^{-2}\{\log(-z)+\gamma\}}{1-e^{-\omega z}}\, dz$$
$$= -\frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{(-z)^{-2}\{\log(-z)+\gamma\}}{1-e^{-\omega z}}\, dz,$$

for it may be readily seen that $\dfrac{\iota}{2\pi}\displaystyle\int_{\frac{1}{\omega}} \frac{\log(-z)+\gamma}{z^2}\, dz = 0.$

[This vanishing contour integral occurs when RAABE's formula is proved by means of the expression of the simple gamma function as a contour integral.]

But as we have deduced in § 75, from ALEXEIEWSKY's theorem,

$$\int_{\omega}^{2\omega} \log \Gamma_2(a\,|\,\omega)\, da - \omega \log \rho_2(\omega) = \omega \log\frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12}.$$

The contour integral has, therefore, the required value.

We here conclude our investigations of the algebra of the double gamma functions. It is evident that the formulæ admit of still further development; they lead, for instance, to many curious relations between the integrals obtained in Part III. Such considerations are, however, foreign to our immediate purpose. The development of the integral formulæ and the theories of multiplication and transformation in the case of the double Riemann ζ function is interesting in that we thus combine many of the formulæ which have been obtained separately for Bernoullian and gamma functions, and the algebra by which such developments are obtained by the extension of MELLIN's definition of the simple ζ function is in many ways attractive. Owing, however, to the length of this paper, we do not propose to consider it in this place.

PART V.

The Asymptotic Expansion and Transcendentally-transcendental Nature of the Double Gamma Function.

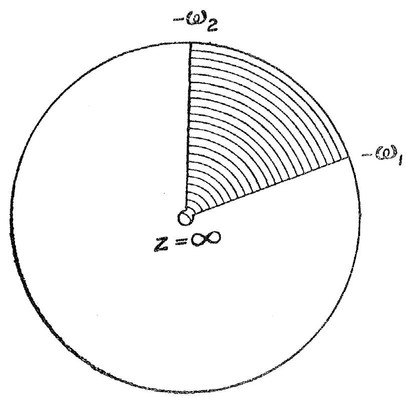
§ 78. There remains now the consideration of two more general characteristics of the double gamma function :—

1. It admits over part of the region near infinity an asymptotic expansion in powers of the variable.
2. It cannot be obtained as the solution of a differential equation whose coefficients involve exclusively more simple functions.

It will afterwards be seen that these characteristics are common to all gamma functions.

Let us consider first the behaviour of $\Gamma_2(z)$ near infinity. We know that its poles are given by

$$z + m_1\omega_1 + m_2\omega_2 = 0 \qquad \left. \begin{array}{l} m_1 = 0, 1, \ldots \infty \\ m_2 = 0, 1, \ldots \infty \end{array} \right\}$$



Therefore near $z = \infty$ the poles of $\Gamma_2(z)$ are massed together between and on the negative axes of ω_1 and ω_2, so as to form a lacunary space on the equivalent portion of the Neumann sphere. Between these axes, therefore, an asymptotic expansion cannot represent the function. We have to consider whether such an expansion can exist outside this lacunary area, within, that is to say, the non-shaded portion of the figure.

We shall in the first place proceed entirely algebraically. It will be proved that within this non-shaded area a quasi-Laurent asymptotic expansion of the form

$$(1, z)^2 \log z + (1, z)^2 + \sum_{r=1}^{\infty} \frac{\lambda_r}{z^r}$$

exists, and then it will be shown that, the possibility of such an expansion being established, its actual form is readily obtained by a process of difference-integration. Subsequently we shall verify the results by an alternative proof by means of contour integration, this proof being the natural extension to double gamma functions of the one employed for functions of a single parameter ("Theory of the Gamma Function," Part IV.).

It should be noticed that the expansion under consideration differs materially from the expansions obtained in Part III. In those expansions the limits of the number of terms of the series and products, the quantities pn and qn, formed the infinite basis terms; but in the present case that basis is the variable itself.

§ 79. We first write down the asymptotic expansion for

$$\log \Gamma_2(z + a \,|\, \omega, \omega).$$

We have obtained ("Theory of the G Function," § 15) the asymptotic expansion

$$\log G(z + a) = \tfrac{1}{12} - \log A + \frac{z + a - 1}{a} \log 2\pi + \left(\frac{(z + a - 1)^2}{2} - \tfrac{1}{12}\right) \log z$$
$$- \frac{3z^2}{4} - (a - 1)z + \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{nz^n} S_n(a \,|\, 1, 1) + \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{2n\,(2n + 2)z^{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-)^n B_n}{2n\,(2n - 1)z^{2n-1}},$$

this expansion being valid for all values of a and z such that $|z|$ is large, and the principal value of $\log z$ being taken. There is, in the language previously employed, a barrier-line along the axis of -1.

Now in § 70 it has been seen that

$$\log G\left(\frac{z}{\omega}\right) = -\log \Gamma_2(z \,|\, \omega) + \frac{z}{2\omega} \log 2\pi - \left(\frac{(z - \omega)^2}{2\omega^2} + \tfrac{1}{2}\right) \log \omega,$$

and
$$\log A - \tfrac{1}{12} = \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \tfrac{1}{2} \log \omega.$$

Therefore

$$-\log \frac{\Gamma_2(z + a \,|\, \omega)}{\rho_2(\omega)} = \left(\frac{(z + a - \omega)^2}{2\omega^2} - \tfrac{1}{12}\right) \log z - \frac{3z^2}{4\omega^2} - z\left(\frac{a}{\omega^2} - \frac{1}{\omega}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{(-)^{-m}}{mz^m} \left\{ {}_2S_m(a \,|\, \omega) + \frac{{}_2S'_{m+1}(o)}{m + 1} \right\},$$

where now there is a barrier-line along the axis of $-\omega$.

But by § 5
$$\frac{{}_2S'_{m+1}(a \,|\, \omega)}{m + 1} = {}_2S_m(a \,|\, \omega) + \frac{{}_2S'_{m+1}(o)}{m + 1},$$

so that we have finally

$$\log \frac{\Gamma_2(z+a\,|\,\omega)}{\rho_2(\omega)} = -\,{}_2S_1'(z+a\,|\,\omega)\log z + \frac{z}{1}\,{}_2S_1^{(2)}(a\,|\,\omega) + \frac{3z^2}{4\omega^2} + \sum_{m=1}^{\infty} \frac{(-)^m\,{}_2S'_{m+1}(a\,|\,\omega)}{m(m+1)z^m},$$

with a barrier-line along the axis of $-\omega$.

When $a = \omega$, we see that we have the asymptotic expansion

$$\log \frac{\Gamma_2(z+\omega\,|\,\omega)}{\rho_2(\omega)} = \left(\frac{1}{12} - \frac{z^2}{2\omega^2}\right)\log z + \frac{3z^2}{4\omega^2} + \sum_{m=1}^{\infty} \frac{(-)^{m-1}\,B_{m+2}(\omega)}{m\omega z^m}.$$

§ 80. We will now prove that, provided z be positive with respect to the ω's, there exists an asymptotic expansion for $\log \Gamma_2(z)$ of the form

$$(1, z)^2 \log z + (1, z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda r}{z^r},$$

where $(1, z)^2$ denotes a quadratic function of z.

In the first place it is evident that we have

$$z = a + n_1\omega_1 + n_2\omega_2,$$

where a is some finite quantity, and n_1 and n_2 are singly or together large positive integers.

Now, from the fundamental difference equations of the double gamma function, it is at once seen that

$$\begin{aligned}\log \Gamma_2(a) - \log \Gamma_2(a + n_1\omega_1 + n_2\omega_2) &= \log \prod_{m_1=0}^{n_1}\prod_{m_2=0}^{n_2}(a + m_1\omega_1 + m_2\omega_2) + \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1\,|\,\omega_2)}{\rho_1(\omega_2)} \\ &+ \log \prod_{m_2=0}^{n_2} \frac{\Gamma_1(a + m_2\omega_2\,|\,\omega_1)}{\rho_1(\omega_1)} - 2m\pi\iota \sum_{m_1=0}^{n_1} S_1'(a + m_1\omega_1\,|\,\omega_2) \\ &- 2m'\pi\iota \sum_{m_2=0}^{n_2} S_1'(a + m_2\omega_2\,|\,\omega_2).\end{aligned}$$

A term has been neglected which is an integral multiple of $2\pi\iota$, and which is therefore absorbed by a suitable specification of the logarithms involved. The above formula may be rewritten

$$\begin{aligned}\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2) &= \left[\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1\,|\,\omega_2)}{\rho_1(\omega_2)}\right] \\ &+ \left[\log \Gamma_2(a) - \log \prod_{m_2=0}^{n_2} \frac{\Gamma_1(a + m_2\omega_2\,|\,\omega_1)}{\rho_1(\omega_1)}\right] \\ &- \left[\log \Gamma_2(a) + \log \prod_{m_1=0}^{n_1}\prod_{m_2=0}^{n_2}(a + m_1\omega_1 + m_2\omega_2)\right] + a(1, n)^2 + (1, n)^2.\end{aligned}$$

In the first place, if we put n_1 in place of pn and n_2 in place of qn in the procedure of the paragraphs leading up to § 49, we evidently obtain for

$$\log \Gamma_2(a) + \log \prod_{m_1=0}^{n_1} \prod_{m_2=0}^{n_2} (a + m_1\omega_1 + m_2\omega_2),$$

an expansion in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, every term of which involves a algebraically, and of which the non-ultimately-vanishing terms are typified by $(1, n)^2 \log n + (1, n)^2$.

In the second place, consider

$$\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1 \mid \omega_2)}{\rho_1(\omega_2)}.$$

We have seen, in § 30, that

$$\Gamma_2(a) = e^{-\gamma_{21}\frac{a^2}{2} - a\left\{\gamma_{22} + \frac{1}{\omega_2}\log\omega_2 - \frac{\gamma}{\omega_2}\right\}} \Gamma_1(a \mid \omega_2)$$
$$\times \prod_{m_1=1}^{\infty} \left[\frac{\Gamma_1(a + m_1\omega_1 \mid \omega_2)}{\Gamma_1(m_1\omega_1 \mid \omega_2)} e^{-a\psi_1^{(1)}(m_1\omega_1 \mid \omega_2) - \frac{a^2}{2}\psi_1^{(2)}(m_1\omega_1 \mid \omega_2)}\right],$$

the product being absolutely convergent.

The typical term may be written

$$\text{Exp.}\left[\frac{a^3}{3!}\psi_1^{(3)}(m_1\omega_1 \mid \omega_2) + \frac{a^4}{4!}\psi_1^{(4)}(m_1\omega_1 \mid \omega_2) + \ldots\right]$$

Therefore ("Genesis of the Double Gamma Function," §§ 4 and 5)

$$\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \Gamma_1(a + m_1\omega_1 \mid \omega_2)$$

admits an asymptotic expansion of the form

$$(1, n_1^2) \log n_1 + (1, n_1)^2 + \sum_{r=1}^{\infty} \frac{\lambda_r}{n_1^r},$$

each term of which involves a algebraically.

Combining these results we see that

$$\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2)$$

admits an asymptotic expansion in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, each term of which involves a algebraically, and of which the terms which do not ultimately vanish are typified by $(1, n)^2 \log n + (1, n)^2$.

But $\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2)$ is a function of $a + n_1\omega_1 + n_2\omega_2$. It must then be capable of an asymptotic expansion in powers of $\frac{1}{n_1\omega_1 + n_2\omega_2}$, each term of which involves a algebraically, and of which the terms which do not ultimately vanish are typified by

$$(1,\ n_1\omega_1 + n_2\omega_2)^2 \log (n_1\omega_1 + n_2\omega_2) + (1,\ n_1\omega_1 + n_2\omega_2)^2.$$

And now by mere re-arrangement we may include a with the term $n_1\omega_1 + n_2\omega_2$ each time that the latter occurs, and we obtain for $\log \Gamma_2(z)$, where $z = a + n_1\omega_1 + n_2\omega_2$, an asymptotic expansion of the form

$$(1,\ z)^2 \log z + (1,\ z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda_r}{z^r}.$$

§ 81. We can readily extend the previous proof to the case where z lies between the axes of $-\omega_1$ and ω_2, so that it is given by

$$z = a - n_1\omega_1 + n_2\omega_2.$$

By writing the fundamental difference equation in the form

$$\Gamma_2(z - \omega_1) = \Gamma_2(z) \frac{\Gamma_1(z - \omega_1 \mid \omega_2)}{\rho_1(\omega_2)}\ e^{-2m\pi\iota S_1'(z - \omega_1 \mid \omega_2)}$$

we readily see that

$$\begin{aligned} &\log \Gamma_2 (z - n_1\omega_1 + n_2\omega_2) \\ &= \left[\log \Gamma_2 (z) - \sum_{m_2=0}^{n_2-1} \log \frac{\Gamma_1(z + m_2\omega_2 \mid \omega_1)}{\rho_1(\omega_1)}\right] \\ &\quad + \left[\log \Gamma_2(z - \omega_1 \mid -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \sum_{m_2=0}^{n_2-1} \log (z - m_1\omega_1 + m_2\omega_2)\right] \\ &\quad + \left[-\log \Gamma_2 (z - \omega_1 \mid -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \log \frac{\Gamma_1(z - m_1\omega_1 \mid \omega_2)}{\rho_1(\omega_2)}\right] \\ &\quad - \sum_{m_1=1}^{n_1} 2m\pi\iota S_1'(z - m_1\omega_1 \mid \omega_2) + \sum_{m_2=0}^{n_2-1} 2m'\pi\iota\, S_1'(z + m_2\omega_2 \mid \omega_1). \end{aligned}$$

But by the theorems just quoted in the previous paragraph the three expressions in the square brackets severally admit of asymptotic expansions in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, whose terms which do not ultimately vanish are typified by

$$(1,\ n)^2 \log n + (1,\ n)^2$$

and whose coefficients all involve a algebraically.

Thus, by a repetition of the previous argument, $\log \Gamma_2(a - n_1\omega_1 + n_2\omega_2)$ admits when $z = a - n_1\omega_1 + n_2\omega_2$, an asymptotic expansion of the form

$$(1,\ z)^2 \log z + (1,\ z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda_r}{z^r}.$$

In an exactly similar manner we may show that $\log \Gamma_2(z)$ will admit of an asymptotic expansion of the same form when $|z|$ is large, z lying between the axes of $-\omega_2$ and ω_1.

§ 82. But when we come to the case of the negative quasi-quadrant given by

$$z = a - n_1\omega_1 - n_2\omega_2,$$

it is interesting to notice that the above proof breaks down.

As before, by the use of the fundamental difference equation, we obtain the relation

$$\log \Gamma_2(a - n_1\omega_1 - n_2\omega_2) - \log \Gamma_2(a) - \log \Gamma_2(a - \omega_1 \mid -\omega_1, \omega_2)$$
$$- \log \Gamma_2(a - \omega_2 \mid \omega_1, -\omega_2) - \log \Gamma_2(a - \omega_1 - \omega_2 \mid -\omega_1, -\omega_2)$$
$$= \left[-\log \Gamma_2(a - \omega_1 \mid -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \log \frac{\Gamma_1(a - m_1\omega_1 \mid \omega_2)}{\rho_1(\omega_2)} \right]$$
$$+ \left[-\log \Gamma_2(a - \omega_2 \mid \omega_2, -\omega_1) + \sum_{m_2=1}^{n_2} \log \frac{\Gamma_1(a - m_2\omega_2 \mid \omega_1)}{\rho_1(\omega_2)} \right]$$
$$- \left[\log \Gamma_2(a - \omega_1 - \omega_2 \mid -\omega_1, -\omega_2) + \sum_{m_2=1}^{n_2} \sum_{m_1=1}^{n_2} \log (a - m_1\omega_1 - m_2\omega_2) \right]$$
$$\left. - \sum_{m_1=1}^{n_1} 2m\pi\iota\, S_1'(a - m_1\omega_1 \mid \omega_2) - \sum_{m_2=1}^{n_2} 2m'\pi\iota\, S_1'(a - m_2\omega_2 \mid \omega_1) \right].$$

The several expressions bracketed on the right-hand side of this identity admit of asymptotic expansions in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, of which the terms involve a algebraically; and therefore the whole of the right-hand side admits of an expansion of this form. But there remain the non-algebraic terms

$$\log \Gamma_2(a) + \log \Gamma_2(a - \omega_1 \mid -\omega_1, \omega_2) + \log \Gamma_2(a - \omega_2 \mid \omega_1, -\omega_2)$$
$$+ \log \Gamma_2(a - \omega_1 - \omega_2 \mid -\omega_1, -\omega_2),$$

and when we seek to group $-n_1\omega_1 - n_2\omega_2$ with a, we are forced back on the original function $\Gamma_2(a - n_1\omega_1 - n_2\omega_2)$. Thus as regards the possibility of an expansion, when z is negative with regard to the ω's, our results are, as we should expect from § 78, entirely negative. The region between the axes of $-\omega_1$ and $-\omega_2$ is a *barrier-region* for the asymptotic expansion of the double gamma function. When $\omega_1 = \omega_2$ this region closes up into the barrier-line which occurs for the G and simple gamma functions.

§ 83. We can now find the asymptotic expansion of

$$\log \frac{\Gamma_2(z + a \mid \omega_1, \omega_2)}{\Gamma_2(z \mid \omega_1, \omega_2)},$$

for large values of $|z|$ which are such that z does not lie in the barrier-region, a being any complex quantity of finite modulus.

For such values of z and a we have the expansion

$$\log \frac{\Gamma_2(z+a)}{\Gamma_2(z)} = [f_1(a)z + f_2(a)] \log z + \phi_1(a)z + \phi_2(a) + \sum_{r=1}^{\infty} \frac{(-)^r \chi_r(a)}{r(r+1)z^r},$$

where $f(a)$ and $\phi(a)$ are algebraical polynomials of degree indicated by their suffixes, and $\chi_r(a)$ is, so long as r is finite, likewise an algebraical function.

Now by the fundamental difference equation

$$\log \frac{\Gamma_2(z+a+\omega_1)}{\Gamma_2(z)} = \log \frac{\Gamma_2(z+a)}{\Gamma_2(z)} - \log \frac{\Gamma_1(z+a\,|\,\omega_2)}{\rho_1(\omega_2)} + 2m\pi\iota\, S_1'(z+a\,|\,\omega_2).$$

Again ("Theory of the Gamma Function," § 41), we have the asymptotic expansion

$$\log \frac{\Gamma_1(z+a\,|\,\omega_2)}{\rho_1(\omega_2)} = S_1'(z+a\,|\,\omega_2)\log_{\omega_2} z - z\,S_1^{(2)}(z+a\,|\,\omega_2) + \sum_{m=1}^{\infty} \frac{(-)^{m+1} S'_{m+1}(a\,|\,\omega_2)}{m(m+1)z^m},$$

where $\log_{\omega_2} z$ is that natural logarithm of z which has its principal value with respect to the axis of $-\omega_2$. It is thus equal to

$$\log z - 2m'\pi\iota,$$

the latter logarithm having its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

We have then, if $\log z$ have its principal value with respect to the axis of $-(\omega_1+\omega_2)$,

$$\begin{aligned}[][f_1(a+\omega_1)z + f_2(a+\omega_1) - f_1(a)z - f_2(a)] \log z \\ + z\{\phi_1(a+\omega_1) - \phi_1(a)\} + \phi_2(a+\omega_1) - \phi_2(a) + \sum_{r=1}^{\infty} (-)^r \frac{\chi_r(a+\omega_1) - \chi_r(a)}{r(r+1)z^r} \\ = -S_1'(z+a\,|\,\omega_2)[\log z - 2(m+m')\pi\iota] + z\,S_1^{(2)}(z+a\,|\,\omega_2) + \sum_{r=1}^{\infty} \frac{(-)^r S'_{r+1}(a\,|\,\omega_2)}{r(r+1)z^r}.\end{aligned}$$

If we equate corresponding powers of z on both sides of this result, we find

$$\chi_r(a+\omega_1) - \chi_r(a) = S'_{r+1}(a\,|\,\omega_2),$$

and similar relations among the f's and ϕ's.

We shall get, in like manner, another set of relations in which ω_1 and ω_2 are interchanged. Remembering that the f's, ψ's, and χ's are all algebraical polynomials which vanish with a, we thus prove that

$$\begin{aligned} f_1(a)z + f_2(a) &= {}_2S_0(z) - {}_2S_0(z+a) \\ \phi_1(a)z + \phi_2(a) &= z\{{}_2S_0'(z+a) - {}_2S_0'(z)\} \\ \chi_r(a) &= {}_2S'_{m+1}(a) - {}_2S'_{m+1}(o). \end{aligned}$$

By this process, which may appropriately be called a process of finite integration, we obtain the asymptotic expansion

$$\log\frac{\Gamma_2(z+a)}{\Gamma_2(z)} = [{}_2S_0(z) - {}_2S_0(z+a)][\log z - 2(m+m')\pi\iota] + z[{}_2S_0'(z+a) - {}_2S_0'(z)] + \sum_{m=1}^{\infty}\frac{(-)^m\,{}_2S_m(a)}{mz^m},$$

$\log z$ having its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

§ 84. We may now obtain the asymptotic expansion for $\log\Gamma_2(z+a)$, under the limitations assigned at the commencement of the preceding paragraph.

For this purpose integrate the relation just obtained with respect to a between the limits o and ω_1.

Then, by the formulæ of §§ 12 and 76, we find

$$-z\log\frac{\Gamma_1(z|\omega_2)}{\rho_1(\omega_2)} + \omega_1[\log\rho_2(\omega_1,\omega_2) - (m+m')2\pi\iota\,{}_2S_1'(o)]$$
$$+(1+2m\pi\iota)\left\{S_1(z|\omega_2)+\frac{\omega_2}{12}\right\} - \omega_2\log\frac{\Gamma_2(z+\omega_2|\omega_2)}{\rho_2(\omega_2)} - \omega_1\log\Gamma_2(z)$$
$$=[{}_2S_1(z) - {}_2S_1(z+\omega_1) + \omega_1\,{}_2S_1'(z)][\log z - 2(m+m')\pi\iota]$$
$$+z[{}_2S_0(z+\omega_1) - {}_2S_0(z) - \omega_1\,{}_2S_0'(z)] + \sum_{m=1}^{\infty}\frac{(-)^m[{}_2S_{m+1}(\omega_1) - \omega_1\,{}_2S'_{m+1}(o)]}{m(m+1)z^m}.$$

Substitute now the asymptotic expansions for $\log\dfrac{\Gamma_1(z|\omega_2)}{\rho_1(\omega_2)}$ and $\log\dfrac{\Gamma_2(z+\omega_2|\omega_2)}{\rho_2(\omega_2)}$, of which the former has been quoted in the preceding paragraph, and the latter obtained in § 79.

Then we find

$$-\omega_1\log\left[\frac{\Gamma_2(z|\omega_1,\omega_2)}{\rho_2(\omega_1\,\omega_2)}e^{(m+m')2\pi\iota\,{}_2S_1'(o\,|\,\omega_1,\,\omega_2)}\right]$$
$$=-(1+2m\pi\iota)\left[S_1(z|\omega_2)+\frac{\omega_2}{12}\right] - \left[S_1(z|\omega_2)+\frac{S_2'(o|\omega_2)}{2} - \omega_1\,{}_2S_1'(z)\right]$$
$$\{\log_{\omega_1+\omega_2}z - 2(m+m')\pi\iota\}$$
$$+z\{S_0(z|\omega_2)+S_1'(0|\omega_2) - \omega_1\,{}_2S_0'(z)\} + z\left(\frac{z}{\omega_2}-\frac{1}{2}\right)\log_{\omega_2}z - \frac{z^2}{\omega_2} + \frac{S_2'(o|\omega_2)}{2}$$
$$+\sum_{n=1}^{\infty}\frac{(-)^n}{(n+1)z^n}\,{}_1B_{n+2}(\omega_2) - \omega_2\log_{\omega_2}z\left\{\frac{z^2}{2\omega_2^2}-\frac{1}{12}\right\} + \frac{3z^2}{4\omega_2}$$
$$+\sum_{n=1}^{\infty}\frac{(-)^{n-1}\,{}_1B_{n+2}(\omega_2)}{nz^n} + \sum_{n=1}^{\infty}(-)^n\frac{{}_1B_{n+2}(\omega_2) - \omega_1\,{}_2S'_{\omega+1}(o)}{n(n+1)z^n}.$$

Remember that $\log_{\omega_2}z = \log_{\omega_1+\omega_2}z - 2m'\pi\iota$; then we obtain by an easy reduction, the asymptotic expansion

$$\log\frac{\Gamma_2(z)\,e^{2\pi\iota(m+m')\,{}_2S_1'(o)}}{\rho_2(\omega_1,\omega_2)}$$
$$=-\,{}_2S_1'(z)\,\{\log_{\omega_1+\omega_2}z - 2(m+m')\pi\iota\} + z\,{}_2S_1^{(2)}(o) + \frac{z^2}{2!}\,{}_2S_1^{(3)}(o)\left(\tfrac{1}{1}+\tfrac{1}{2}\right)$$
$$+\sum_{m=1}^{\infty}\frac{(-)^m\,{}_2S'_{m+1}(o)}{m(m+1)z^m},$$

which is the complete asymptotic expansion for $\log \Gamma_2(z)$ when $|z|$ is large, and z does not lie within the barrier region negative with respect to the axes of $-\omega_1$ and $-\omega_2$.

If we combine this result with that obtained in § 83 we find the more general formula

$$\log \frac{\Gamma_2(z+a)\, e^{2\pi\iota(m+m')\, {}_2S_1'(o)}}{\rho_2(\omega_1, \omega_2)}$$
$$= -{}_2S_1'(z+a)\{\log_{\omega_1+\omega_2} z - 2(m+m')\pi\iota\} + z\,{}_2S_1^{(2)}(a) + \frac{z_2}{2!}\,{}_2S_1^{(3)}(a)\left(\tfrac{1}{1}+\tfrac{1}{2}\right)$$
$$+ \sum_{m=1}^{a} \frac{(-)^m\, {}_2S'_{m+1}(a)}{m(m+1)\, z^m},$$

valid under the assigned limitations.

The expansion is written in the precise form adopted, in order that the analogy with the corresponding formula in the theory of multiple gamma functions may be more clearly displayed.

§ 85. We might now conclude this investigation. Since, however, this would appear to be the first time in analysis in which an asymptotic expansion with a barrier region has been obtained, it seems better to give an alternative proof which shall not need the difficult argument of §§ 80–82. This proof is the direct extension of that previously given for the case of the simple gamma function.* We therefore proceed as briefly as possible.

In the investigation of § 57 it was shown that when $|s|$ is finite and $\mathfrak{R}(s) > -k$, where k is a positive integer, the series for $\zeta_2(s, a\,|\,\omega_1, \omega_2)$ is absolutely convergent.

Suppose now that $s = \sigma + i\tau$, where $\sigma > -k$, and suppose further that z does not lie within the region bounded by axes to $-\omega_1$ and $-\omega_2$, and that a is positive with respect to the ω's.

Then, since

$${}_2S_{-s,k}(a) = \frac{1}{(1-s)(2-s)\,\omega_1\,\omega_2\, a^{s-2}} - \frac{\omega_1+\omega_2}{2(1-s)\,\omega_1\,\omega_2\, a^{s-1}} + \frac{{}_2B_1}{a^s} + \sum_{r=0}^{k-1}\binom{-s}{r}\frac{{}_2B_{r+1}}{a^{s+r}},$$

it is evident that, if p be any positive integer, the absolute value of each term of the expression

$$\frac{s^p z^s}{\sin \pi s}\, {}_2S_{-s,k}\,(a\,|\,\omega_1, \omega_2)$$

tends to zero as $|\tau|$ tends to infinity. For, by the restrictions on z and a,

$$\frac{z^s}{a^s} = (r\, e^{\iota\psi})^{\sigma + \iota\tau}, \qquad \text{where } o < \psi < \pm\pi,$$

and therefore

* "Theory of the Gamma Function," Part IV. I regret to say that the Lemma of § 40 is faulty; the theorem is evidently only true when a/ω is real. A slight modification will, however, establish the truth of the main proposition under the conditions enunciated.

$$\left| \frac{z^s}{a^{s+l}} \frac{s^p}{\sin \pi s} \right| = e^{-\psi\tau - \pi|\tau|} \cdot \tau^p \text{M},$$

where M is finite, however large $|\tau|$ may be.

Hence the absolute value of

$$\frac{s^p z^s}{\sin \pi s} {}_2\text{S}_{-s,k}(a \mid \omega_1, \omega_2)$$

tends to zero as $|\tau|$ tends to infinity; and this theorem is true if a is replaced by $a + m_1\omega_1 + m_1\omega_2$, where m_1 and m_2 are positive integers.

Hence the absolute value of

$$\frac{s^p z^s}{\sin \pi s} \chi(a + m_1\omega_1 + m_2\omega_2 \mid s, k)$$

(where χ is the function introduced for brevity in § 57) tends to zero as $|\tau|$ tends to infinity.

But

$$\zeta_2(s, a \mid \omega_1, \omega_2) = {}_2\text{S}_{-s,k}(a) - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \chi(a + m_1\omega_1 + m_2\omega_2 \mid s, k),$$

and therefore

$$\frac{s^p z^s}{\sin \pi s} \zeta_2(s, a \mid \omega_1, \omega_2) = \frac{s^p z^s {}_2\text{S}_{-s,k}(a)}{\sin \pi s} - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{s^p z^s}{\sin \pi s} \chi(a + m_1\omega_1 + m_2\omega_2 \mid s, k).$$

Now the double series on the right-hand side is absolutely convergent for all finite values of $|\tau|$, and the absolute value of each term tends to zero as $|\tau|$ tends to infinity.

Therefore

$$\left| \frac{s^p z^s \zeta_2(s, a \mid \omega_1, \omega_2)}{\sin \pi s} \right|$$

remains finite as $|s|$ tends to infinity, $\mathfrak{R}(s)$ being finite and not greater than 2.

When $\mathfrak{R}(s)$ is greater than 2, we have

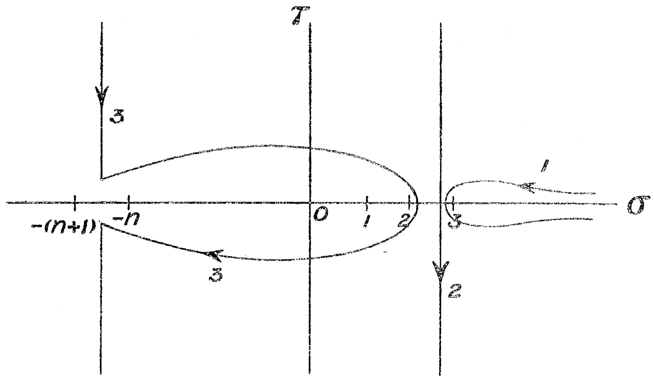
$$\left| \frac{s^p z^s \zeta_2(s, a \mid \omega_1, \omega_2)}{\sin \pi s} \right| = \left| \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{z^s s^p}{(a + m_1\omega_1 + m_2\omega_2)^s \sin \pi s} \right|,$$

and therefore the expansion on the left-hand side is finite however large $|s|$ may be, provided $|z| < 1$.

§ 86. Consider now the integral

$$\frac{1}{2\pi\iota} \int z^s \cdot \frac{\pi \zeta_2(s, a \mid \omega_1, \omega_2)}{s \sin \pi s} ds.$$

The subject of integration is a uniform function of s, wherein z^s is to have its principal value with respect to the axis of $-(\omega_1 + \omega_2)$, z is to lie within the region bounded by axes to $-\omega_1$ and ω_2, and a is to be positive with respect to the ω's.



In the first place let the contour be taken to lie along the real axis, passing from $+\infty$ to $+\infty$, and cutting the axis between the points $\sigma = 2$ and $\sigma = 3$, as the contour 1 of the figure. This is equivalent to taking the integral round a contour enclosing the points 3 and $+\infty$.

When $|z| < 1$ the integral is, by the theorem of the preceding paragraph, finite; and by CAUCHY's theorem it will be equal in value to the sum of the residues inside the contour.

Now by § 53, when $s = 2 + k$, where k is an integer,

$$\zeta_2(s, a) = \frac{(-)^k}{(k+1)!} \psi_2^{(2+k)}(a).$$

Hence the value of the integral along the contour 1 is

$$\sum_{k=0}^{\infty} \frac{z^{2+k}}{(k+2)!} \frac{d^{2+k}}{da^{2+k}} \log \Gamma_2(a),$$

and by TAYLOR's theorem this expression is, under the assigned limitations, equal to

$$\log \frac{\Gamma_2(z+a)}{\Gamma_2(a)} - z\psi_2^{(1)}(a).$$

Let us now make the contour expand until it becomes a straight line perpendicular to the axis of σ, cutting the axis between the points 2 and 3, and a half circle at infinity. The value of the integral will be unaltered, since the contour in expanding passes over no poles of the subject of integration. And by the theorem of the previous paragraph the part of the integral which is taken along the semicircle at infinity vanishes. Hence the integral along the perpendicular line (the contour numbered 2 in the figure) is equal to

$$\log \frac{\Gamma_2(z+a)}{\Gamma_2(a)} - z\psi_2^{(1)}(a), \text{ when } |z| < 1.$$

But the integral and this expression both remain continuously finite when $|z|$ becomes greater than unity. They are therefore equal to one another for all values of $|z|$.

Let now the perpendicular contour be distorted into a contour which encloses the points 2, 1, 0, . . . $-n$, and which after the point $\sigma = -n$ again goes off to infinity perpendicularly to the real axis.

So far as the value of the integral is concerned this contour will differ from the second contour only by two strips at infinity of length less than $(n+3)$ parallel to the real axis: and by the previous paragraph the integral along these strips will vanish.

By CAUCHY's theorem the value of the integral along this third contour will be equal to minus the sum of its residues at the points 2, 1, 0, . . . $-n$, together with the integral along the perpendicular line cutting the axis of σ between the points $-n$ and $-(n+1)$.

Now, when $s = 1$ or 2, the residue of the integral is equal to the coefficient of $1/\epsilon$ in

$$\frac{(-)^s}{\epsilon}\left(1 - \frac{\epsilon}{s} + \ldots\right)\left\{\frac{(-)^{s+1}}{\epsilon(s-1)!}{}_2S_1^{(s+1)}(a) + \frac{(-)^{s+1}}{(s-1)!}\left(\frac{1}{} + \ldots + \frac{1}{s-1}\right){}_2S_1^{(s+1)}(a)\right.$$
$$\left. + \frac{(-)^s}{(s+1)!}\psi_2^{(s)}(a) + (-)^{s-1}2(m+m')\pi\iota\,{}_2S_1^{(s+1)}(a)\right\}\frac{z^s}{s}\{1 + \epsilon\log z + \ldots\}$$

by § 53,

and is therefore equal to

$$\frac{-z^s}{s!}\left[\frac{1}{1} + \ldots + \frac{1}{s} - \log z\right]{}_2S_1^{(s+1)}(a) + \frac{z^s}{s!}[\psi_2^{(s)}(a) - 2(m+m')\pi\iota\,{}_2S_1^{(s+1)}(a)],$$

where the logarithm has its principal value with respect to the axes of $-(\omega_1 + \omega_2)$.

When $\epsilon = 0$, the residue is the coefficient of $1/\epsilon$ in

$$\frac{1 + \epsilon\log z + \ldots}{\epsilon^2}\left\{{}_2S'_1(a) + \epsilon\log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}e^{-{}_2S_0(a)2(m+m')\pi\iota}\right\} \qquad \text{by § 60,}$$

and is therefore equal to

$${}_2S'_1(a)\log z + \log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}e^{-{}_2S_0(a)2(m+m')\pi\iota}.$$

When $s = -m$, the residue is

$$\frac{(-)^{m+1}{}_2S_{m+1}(a)}{m(m+1)z^m}.$$

We therefore have

$$\log\frac{\Gamma_2(z+a)}{\Gamma_2(a)} - \frac{z}{1!}\psi_2^{(1)}(a)$$
$$= -\log\left\{\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}e^{{}_2S'_1(o)2(m+m')\pi\iota}\right\} - {}_2S'_1(a)[\log z - 2(m+m')\pi\iota]$$
$$- \sum_{s=1}^{2}\frac{z^s}{s!}\left[\psi_2^{(s)}(a) + {}_2S_1^{(s+1)}(a)\left\{\log z - 2(m+m')\pi\iota - \frac{1}{1} - \ldots - \frac{1}{s}\right\}\right]$$
$$+ \sum_{m=1}^{n}\frac{(-)^m{}_2S'_{m+1}(a)}{m(m+1)z^m} + J_n(z, a\,|\,\omega_1, \omega_2)$$

where the quantity $\log z$ has its principal value with respect to the axis of $-(\omega_1 + \omega_2)$ and $J_n(z, a \mid \omega_1, \omega_2)$ is equal to the fundamental integral taken along a perpendicular contour cutting the axis of σ between $-n$ and $-(n+1)$. It is evident that the integral when $|z|$ is large is of an order of magnitude less than $\dfrac{1}{|z|^n}$.

We therefore have the asymptotic expansion, when $|z|$ is large,

$$\log \frac{\Gamma_2(z+a)\, e^{2\pi\iota(m+m')_2S_1'(o)}}{\rho_2(\omega_1, \omega_2)} = -\frac{z^2}{2!}\, {}_2S_1^{(3)}(a)\left[\log z - 2(m+m')\pi\iota - \tfrac{1}{1} - \tfrac{1}{2}\right]$$

$$-\frac{z}{2!}\, {}_2S_1^{(2)}(a)\left[\log z - 2(m+m')\pi\iota - \tfrac{1}{1}\right]$$

$$-\,{}_2S_1'(a)\left[\log z - 2(m+m')\pi\iota\right] + \sum_{m=1}^{\infty} \frac{(-)^m\, {}_2S'_{m+1}(a)}{m(m+1)\, z^m}$$

and the residue after n terms of the final series have been taken is of the same order of magnitude as the final term taken.

This expansion is evidently the same as that previously obtained. The limitation that a must be positive with respect to the ω's may evidently be removed by employing the fundamental difference relations for the double gamma function and the asymptotic expansion for $\log \Gamma_1(z+a)$. We are finally left with the essential limitation that z shall not lie within the barrier region bounded by the axis to $-\omega_1$ and $-\omega_2$.

The Transcendentally-transcendental Nature of $\Gamma_2(z)$.

§ 87. We finally prove the theorem that the double gamma function cannot arise as the solution of a differential equation whose coefficients are not generated from the function itself. Modifying slightly the nomenclature introduced by MOORE,* we may say that $\Gamma_2(z)$ is a transcendentally-transcendental function. The proof is a slight modification of that given for the G function (§ 30), which in turn was similar to the investigation of Part V. of the "Theory of the Gamma Function."

In the first place it may be proved exactly as before that if the theorem is true for $\dfrac{d^2}{dz^2} \log \Gamma_2(z)$, it is true for $\Gamma_2(z)$. We shall therefore confine ourselves to the consideration of the function

$$\phi(z) = -\frac{d^2}{dz^2} \log \Gamma_2(z).$$

By the fundamental difference equations of § 20, we have

$$\phi(z + \omega_1) - \phi(z) = \psi(z \mid \omega_2)$$
$$\phi(z + \omega_2) - \phi(z) = \psi(z \mid \omega_1)$$

where, for convenience, we put $\psi(z) = \dfrac{d^2}{dz^2} \log \Gamma_1(z)$.

* MOORE, 'Math. Ann.,' vol. 48, pp. 49 *et seq.* MOORE uses the term only to describe functions which cannot be generated by a differential equation with algebraic coefficients.

Suppose that

$$y = \phi(x) \text{ satisfies the differential equation}$$
$$f(x, y, y', \ldots y^{(n)}) = 0,$$

so transformed that it is rational and integral in y and its derivatives.

Let the terms of class s be symbolically

$$R_0(x)\, Q_s^0(y),\ R_1(x)\, Q'_s(y),\ \ldots\ ,\ R_k(x)\, Q_s^k(y);$$

in terms of class $(s-1)$ being

$$S_0(x)\, Q^0_{s-1}(y),\ \ldots\ldots\ldots\ ,\ S_l(x)\, Q^l_{s-1}(y),$$

and the functions $R(x)$, $S(x)$ being holomorphic.

If $\phi(x)$ satisfies the differential equation, $\phi(x) + \psi(x|\omega_2)$ will satisfy the equation in which $(x+\omega_1)$ is written for x and $\phi(x) + \psi(x|\omega_1)$ the equation in which $(x + \omega_2)$ is written for x.

Make the first substitution, divide the equations by $R_0(x)$ and $R_0(x + \omega_1)$ respectively, and subtract one from the other. We find

$$\frac{R_1(x+\omega_1)}{R_0(x+\omega_1)} Q'_s[\phi(x) + \psi(x|\omega_2)] + \ldots + \frac{R_k(x+\omega_1)}{R_0(x+\omega_1)} Q_s^k[\phi(x) + \psi(x|\omega_2)]$$
$$-\left\{\frac{R_1(x)}{R_0(x)} Q_s'[\phi(x)] + \ldots + \frac{R_k(x)}{R_0(x)} Q_s^k[\phi(x)]\right\} + Q_s^0[\phi(x) + \psi(x|\omega_2)] - Q_s^0[\phi(x)]$$
$$+ \text{ terms of lower class} = 0\,.$$

But

$$Q_s^0[\phi(x) + \psi(x|\omega_2)] - Q_s^0[\phi(x)]$$

consists solely of terms of lower class than s.

Hence either the equation which has been obtained vanishes identically, or we can reduce the equation for y to one in which there are fewer terms of class s.

The equation cannot vanish identically unless the coefficients of the various terms of class s all vanish, which necessitates that the ratios

$$\frac{R_1(x)}{R_0(x)},\ \ldots\ ,\ \frac{R_k(x)}{R_0(x)}$$

are doubly periodic functions of x of periods ω_1 and ω_2.

The equation for y can thus be always reduced to one of the form

$$R(x)\,[p_0(x)\, Q_s^0(y) + \ldots + p_k(x) Q_s^k(y)]$$
$$+ S_0(x) Q_{s-1}^0(y) + \ldots + S_l(x) Q_{s-1}^l(y)$$
$$+ \text{ terms of lower class} = 0.$$

where all the coefficients are holomorphic functions, and, in addition, the functions $p(x)$ are doubly periodic of periods ω_1 and ω_2.

Divide the equation by $R(x)$ and subtract it from the equation which results from changing x into $x + \omega_1$. We obtain

$$p_0(x)[Q_s^0\{\phi(x) + \psi(x|\omega_2)\} - Q_s^0\{\phi(x)\}] + \ldots + p_k(x)[Q_s^k\{\phi(x) + \psi(x|\omega_2)\} - Q_s^k\{\phi(x)\}]$$

$$+ \frac{S_0(x + \omega_1)}{R(x + \omega_1} Q_{s-1}^0\{\phi(x) + \psi(x|\omega_2)\} - \frac{S_0(x)}{R(x)} Q_{s-1}^0\{\phi(x)\}$$

$$+ \ldots\ldots\ldots$$

$$+ \frac{S_l(x + \omega_1)}{R(x + \omega_1)} Q_{s-1}^l\{\phi(x) + \psi(x|\omega_2)\} - \frac{S_l(x)}{R(x)} Q_{s-1}^l\{\phi(x)\}$$

$$+ \text{terms of lower class} = 0.$$

This equation will not vanish identically unless the functions

$\frac{S_l(x)}{R(x)}, \ldots \frac{S_l(x)}{R(x)}$ all satisfy relations of the form

$$f(x + \omega_1) - f(x) = \sum_{k=0}^{n} p_k(x)\, Q_1^k[\psi(x|\omega_2)]$$

and therefore by symmetry relations of the form

$$f(x + \omega_2) - f(x) = \sum_{k=0}^{n} p_k(x)\, Q_1^k\left[\psi(x|\omega_1)\right]$$

The quantity $(n + 1)$ which gives the number of terms on the right-hand side of these two relations will not vanish unless the original equation can be reduced to the form

$$\begin{aligned} R(x)[p_{1,0}(x)\, Q_s^0(y) + \ldots + p_{1,k}(x)\, Q_s^k(y) \\ + p_{2,0}(x)\, Q_{s-1}^0(y) + \ldots + p_{2,l}(x)\, Q_{s-1}^l(y)] \\ + T_0(x)\, Q_{s-2}^0(y) + \ldots + T_m(x)\, Q_{s-2}^{(m)}(y) \\ + \text{terms of lower class} = 0 \quad \ldots\ldots\ldots \quad (1), \end{aligned}$$

where the coefficients are holomorphic functions of x and the p's are doubly periodic functions of periods ω_1 and ω_2.

Either then the original equation can be reduced to this form, or at least one of the ratios $\frac{S_0(x)}{R(x)}, \ldots, \frac{S_l(x)}{R(x)}$ is composed of an additive number of solutions of difference equations of the form

$$\begin{cases} f(x + \omega_1) - f(x) = p(x)\, \psi(x|\omega_2) \\ f(x + \omega_2) - f(x) = p(x)\, \psi(x|\omega_1) \end{cases}$$

But the most general solution of such a pair of equations is

$$q(x)\, \phi(x) + r(x),$$

where $q(x)$ and $r(x)$ are doubly periodic functions of x of periods ω_1 and ω_2. And therefore one of the ratios $\frac{S_0(x)}{R(x)}, \ldots, \frac{S_l(x)}{R(x)}$ must be a function generated from the function $\Gamma_2(x \mid \omega_1, \omega_2)$.

The original equation therefore either contains the double gamma function implicitly among its coefficients, or it is reducible to the form (1).

Continue our former procedure, and we see that either at least one of the ratios $\frac{T_0(x)}{R(x)}, \ldots \frac{T_m(x)}{R(x)}$ is composed of an additive number of equations of the type

$$f(x+\omega_1) - f(x) = p(x)\, Q_2 \{\psi(x \mid \omega_2)\}$$
$$f(x+\omega_2) - f(x) = p(x)\, Q_2 \{\psi(x \mid \omega_1)\}$$

and is therefore generated from the double gamma function, or the original equation is reducible to one in which the ratios of terms of the three highest classes are doubly periodic functions of x of periods ω_1 and ω_2.

The successive repetitions of the argument are now evident. Ultimately we reduce the equation to one in which either all the coefficients are doubly periodic functions (which is absurd), or to one in which the last term is generated from the double gamma function.

Thus the proposition is established. The double gamma function cannot satisfy a differential equation in which the coefficients are finite combinations of, *e.g.*,

(1) Rational or irrational algebraic functions of x,
(2) Simply or doubly periodic functions,
(3) Simple gamma functions,
(4) G functions,
(5) Theta functions,

or, in fact, of any functions which are not substantially reducible to or compounded of the double gamma function itself.